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# Variance swap with mean reversion, multifactor stochastic volatility and jumps

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ABSTRACT

This paper examines variance swap pricing using a model that integrates three major features of financial assets, namely the mean reversion in asset price, multi-factor stochastic volatility (SV) and simultaneous jumps in prices and volatility factors. Closed-form solutions are derived for vanilla variance swaps and gamma swaps while the solutions for corridor variance swaps and conditional variance swaps are expressed in a one-dimensional Fourier integral. The numerical tests confirm that the derived solution is accurate and efficient. Furthermore, empirical studies have shown that multi-factor SV models better capture the implied volatility surface from option data. The empirical results of this paper also show that the additional volatility factor contributes significantly to the price of variance swaps. Hence, the results favor multi-factor SV models for pricing variance swaps consistent with the implied volatility surface.



## 1. Introduction

The management of volatility risk has gained increased attention in financial markets since the onset of the recent global financial crisis. Variance swap is a typical financial tool for managing this risk. Numerous researches have been done on variance swaps (Carr & Madan, 1998; Demeterfi, Derman, Kamal, & Zou, 1999). However, none of them consider the discrete monitoring principle or the use of stochastic volatility model. Until recently, Zhu and Lian (2011) solve the discretely sampled variance swap pricing formula under the Heston's stochastic volatility (SV) model using a partial differential equation (PDE) approach. Zheng and Kwok (2012) consider the stochastic volatility simultaneous jump (SVSJ) model in the valuation of various types of variance swap contracts using a probabilistic approach. They also include saddle-point approximation in Zheng and Kwok (2013a) and Fourier transform algorithms in Zheng and Kwok (2013b), respectively, in variance swap pricing under Lévy processes.

In this paper, we generalize these recent advances in variance swap pricing to a wider class of models that incorporates the following well-known features of financial asset dynamics: mean reversion in asset price, multi-factor SV and simultaneous jumps in price and volatility factors. Therefore, the model considered embraces the Heston SV and SVSJ models as its special cases. Mean reversion is a well-known feature in commodity markets. A list of empirical studies supporting the existence of mean reversion can be found in Fusai, Marena, and Roncoroni (2008). Their model is extended to incorporate jumps by Chung and Wong (2014). Wong and Lo (2009) propose an option pricing model with mean reversion and the Heston SV to capture information contained in the term structure of futures prices. The Wong and Lo model is also found to be a special case of our model.

Another well-known empirical finding is that the implied volatility surface solved by matching market option prices and the Black–Scholes formula shows a smiling pattern. Numerous models have been proposed to capture this pattern, including the SV models. Empirical evidence, however, shows that the level of implied volatility is independent of the slope of the volatility smile. Although a one-factor SV model can generate a steep smile or a flat smile for a given volatility, it fails to generate both patterns for a given parameterization. Using the Black–Scholes implied volatility for S&P 500 options over 15 years, the empirical study by Christoffersen, Heston, and Jacobs (2009) shows that a two-factor SV model is sufficient to model the stock return volatility based on a principal component analysis. The empirical study by Li and Zhang (2010) confirms the existence of the second volatility factor using a nonparametric test. Therefore, we consider a two-factor SV model in this paper.

Apart from mean reversion and multifactor SV, jumps in the return process and volatility factors have also received a great deal of research attention. Jacod and Todorov (2010) discover that the prices of most of the constituent stocks in the S&P 500 index jump together with their volatilities. Duffie, Pan, and Singleton (2000) compare an SV model without jumps, an SV model with jump in price and an SVSJ model and show that the SVSJ model is able to produce an implied volatility smile closest to the one observed in the market. Wong and

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Zhao (2010) apply a model with mean reversion and two SV factors to currency option pricing. However, they do not incorporate jumps nor investigate the implications of their model for variance swap pricing.

In this paper, we value various types of discretely monitored variance swaps under the proposed model. We follow Zheng and Kwok (2012) in using the square of the geometric return of the underlying asset to represent the realized variance in the variance swaps. The solution for the fair strike price in the variance swap contract, or the variance swap price, relies on the analytic joint characteristic function of the log asset prices at two different time points. We obtain this characteristic function by first deriving the joint characteristic function of the log asset price and its volatility at a particular time point. Then, we transform the payoff function of the variance swap into an exponential function. This enables the variance swap price to be deduced from the characteristic function.

The remainder of this paper is organized as follows. The proposed model is presented in Section 2, where we also derive the joint characteristic function of the log asset price and the volatility. The analytical formulas for various types of variance swap contracts are obtained in Section 3. Empirical and numerical experiments are conducted in Section 4. Specifically, the empirical experiments illustrate the effect of the second volatility factor on variance swap prices. Section 5 concludes the paper.

## 2. The model

Under the risk-neutral measure, we postulate that the underlying asset $S_t$ and its two volatility factors $V_{1t}$ and $V_{2t}$ jointly evolve as follows. Let $S_t = \exp(X_t)$.

$$\begin{aligned} dX_t &= [\theta(t) - mX_t - \frac{V_{1t}+V_{2t}}{2}]dt + \sqrt{V_{1t}}dW_t^1 + \sqrt{V_{2t}}dW_t^3 + J_t^X dN_t^1,\\ dV_{1t} &= [a_1(t) - b_1 V_{1t}]dt + \sigma_1\sqrt{V_{1t}}[\rho_1 dW_t^1 + \sqrt{1-\rho_1^2}dW_t^2] + J_t^{V_1}dN_t^1,\\ dV_{2t} &= [a_2(t) - b_2 V_{2t}]dt + \sigma_2\sqrt{V_{2t}}[\rho_2 dW_t^3 + \sqrt{1-\rho_2^2}dW_t^4] + J_t^{V_2}dN_t^2, \end{aligned} \tag{1}$$

where $W_t^1$, $W_t^2$, $W_t^3$ and $W_t^4$ are independent standard Wiener processes; $N_t^1$ and $N_t^2$ are independent Poisson processes with constant intensities $\lambda_1$ and $\lambda_2$, respectively; and $J^{V_2}$, $J^{V_1}$ and $J^X$ are independent random variables that represent random jump sizes and which are independent of the Wiener processes and Poisson processes.

This model (1) embraces most of the important derivatives pricing models in the literature. The simultaneous jumps on asset return and its volatility considered by Zheng and Kwok (2012) are reflected by the Poisson process $N_t^1$, while $N_t^2$ models jumps in the volatility process independent of the asset return. The constant $m$ is the mean-reversion speed of the log asset, the deterministic function $\theta(t)$ is related to the equilibrium mean level of the log asset at time $t$ and $J^X$ denotes the random jump size of the log asset. Similarly, the constant $b_i$ is the mean-reversion speed of the $i$th volatility factor, the deterministic function $a_i(t)$ is related to the equilibrium mean level of the $i$th volatility factor at time $t$, the constant $\sigma_i$ is the volatility coefficient of the $i$th volatility factor process and $J^{V_i}$ denotes the random jump size of the $i$th volatility factor for $i = 1, 2$. Notice that it is not necessary to know the explicit form of $\theta(t)$ since we will show in later part that this term can be calibrated with futures as a whole. If there is no jump, the model is reduced to the mean reversion model with two-factor SV in Wong and Zhao (2010). If the mean-reversion speed $m$ is further set to zero, it becomes the two-factor SV model proposed by Christoffersen et al. (2009).

### 2.1. The characteristic function

The payoff of variance swaps depends on the underlying asset prices realized at time points $0 < t_1 < \cdots < t_n$. Thus, the valuation needs the joint distribution of asset prices at these time points. We thus derive the multivariate characteristic function of log-spot prices, $\ln S_{t_1}, \ldots, \ln S_{t_n}$, under the proposed model and then apply the results to the variance swap pricing in the next section.

This section is organized as follows. We begin with a mean reversion model with a one-factor SV and extend it to incorporate the second SV factor. As shown in the later analysis, the process with two SV factors can be written as the sum of two independent processes with one SV factor. As the building block for the later analysis, the following lemma presents the joint characteristic function of the log-asset value and its variance under the mean reversion model with a one-factor SV process and simultaneous jumps. Afterward we derive the joint characteristic function of the log-asset values under the model (1).

**Lemma 2.1.** *Consider the mean reversion model with SV and simultaneous jumps:*

$$\begin{aligned} dX_t &= [\theta(t) - mX_t - \tfrac{V_t}{2}]dt + \sqrt{V_t}dW_t^1 + J_t^X dN_t,\\ dV_t &= [a(t) - bV_t]dt + \sigma\sqrt{V_t}[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2] + J_t^V dN_t, \end{aligned} \tag{2}$$

*where $W_t^1$ and $W_t^2$ are independent standard Wiener processes; $N_t$ is a Poisson process with constant intensity $\lambda$ independent of the two Wiener processes; and $J_t^X$ and $J_t^V$ are random jump sizes of the log asset price and volatility, respectively. $J_t^X$ and $J_t^V$ are independent of the two Wiener processes and the Poisson process. Then, the joint characteristic function of $X_t$ and $V_t$ is given by*

$$f(x, v, t; \phi, \varphi) = \mathbb{E}[\exp(i\phi X_T + i\varphi V_T)|X_t = x, V_t = v] \tag{3}$$

$$= \exp[A(T-t;\phi)x + B(T-t;\phi,\varphi)v + C(T-t;\phi,\varphi)], \tag{4}$$

*where $T \geq t$ and $i = \sqrt{-1}$,*

$$A(\tau;\phi) = i\phi e^{-m\tau},$$

$$B(\tau;\phi,\varphi) = U(e^{-m\tau}) + \frac{e^{-b\tau}V(e^{-m\tau})}{\frac{1}{i\varphi - U(1)} + \frac{\sigma^2}{2m}\int_1^{e^{-m\tau}} y^{\frac{b}{m}-1}V(y)dy},$$

$$\begin{aligned} C(\tau;\phi,\varphi) = \int_{T-\tau}^{T} &[\theta(s)A(T-s;\phi) + a(s)B(T-s;\phi,\varphi)\\ &+ \lambda\mathbb{E}[\exp(A(T-s;\phi)J^X + B(T-s;\phi,\varphi)J^V) - 1]]ds, \end{aligned}$$

$$U(y;\phi) = \frac{2my}{\sigma^2}\frac{(\sqrt{1-\rho^2} - \rho i)\frac{\sigma\phi}{2m}\Phi(a^*, b^*, \frac{y}{\omega(\phi)}) + \frac{a^*}{b^*\omega(\phi)}\Phi(a^*+1, b^*+1, \frac{y}{\omega(\phi)})}{\Phi(a^*, b^*, \frac{y}{\omega(\phi)})},$$

$$V(y;\phi) = \frac{\Phi^2(a^*, b^*, \frac{1}{\omega(\phi)})}{\Phi^2(a^*, b^*, \frac{y}{\omega(\phi)})}e^{\frac{\sigma\phi}{m}(1-y)\sqrt{1-\rho^2}},$$

$$a^* = \frac{(\sqrt{\rho^2-1}+\rho)\frac{b^*}{2} + \frac{\sigma}{4m}}{\sqrt{\rho^2-1}}, \quad b^* = 1 - \frac{b}{m}, \quad \omega(\phi) = \frac{-m}{\sigma\phi\sqrt{1-\rho^2}},$$

*and $\Phi(\cdot,\cdot,\cdot)$ is the degenerated hypergeometric function.*

**Proof.** The Feynman–Kac formula states that $f(x, v, t)$ is governed by the following partial integro-differential equation (PIDE):

$$\begin{cases} -\frac{\partial f}{\partial t} = [\theta(t) - mx - \frac{v}{2}]\frac{\partial f}{\partial x} + [a(t) - bv]\frac{\partial f}{\partial v}\\ \qquad + \frac{v}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\sigma^2 v}{2}\frac{\partial^2 f}{\partial v^2} + \rho\sigma v\frac{\partial^2 f}{\partial x\partial v}\\ \qquad + \lambda\mathbb{E}[f(x+J^X, v+J^V, t) - f(x, v, t)],\\ f(x, v, T) = \exp(i\phi x + i\varphi v). \end{cases}$$

From the affine structure of our model, we postulate $f(x, v, t)$ admitting the form (4). Substituting (4) into the above PIDE gives the following system of ordinary differential equations for $A$, $B$ and $C$:

$$\frac{\partial A}{\partial\tau} = -mA(\tau),$$