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Decision Support

Coalitional multinomial probabilistic values ☆

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ABSTRACT

We introduce a new family of coalitional values designed to take into account players' attitudes with regard to cooperation. This new family of values applies to cooperative games with a coalition structure by combining the Shapley value and the multinomial probabilistic values, thus generalizing the symmetric coalitional binomial semivalues. Besides an axiomatic characterization, a computational procedure is provided in terms of the multilinear extension of the game and an application to the Catalonia Parliament, Legislature 2003–2007, is shown.

1. Introduction

Value theory started by Shapley (1953), who introduced the axiomatic method in cooperative game theory to define a solution concept called now the *Shapley value*. The Shapley value is characterized as the unique value to satisfy efficiency, the null player property, symmetry, and additivity.

Weber (1988) obtained a wide generalization of the Shapley value by defining the family of *probabilistic values*, each one of which requires weighting coefficients p_S^i for each player i and each coalition $S \subseteq N\backslash\{i\}$. The payoff that a probabilistic value allocates to each player is a weighted sum of his marginal contributions in the game. We quote from Weber (1988):

> "Let player i view his participation in a game v as consisting merely of joining some coalition S and then receiving as a reward his marginal contribution to the coalition. If p_S^i is the probability that he joins coalition S, then $\phi_i[v]$ is his expected payoff from the game."

Puente (2000) (see also Freixas & Puente, 2002) defined two special subfamilies of probabilistic values: (a) *binomial semivalues*, where the weighting coefficients depend on a unique parameter $q \in [0, 1]$; and (b) *multinomial probabilistic values*, where the weighting coefficients depend on n parameters, one per player and all in $[0, 1]$ too.[1] Of course, (a) is a subfamily of (b).

Introduced by Aumann and Drèze (1974), the notion of *game with a coalition structure* provided a new avenue for the development of value theory. Since then, a lot of work has been done in this new field, mainly addressed to define *coalitional* values that often represent extensions of classical values to this setup. The best known coalitional value is the *Owen value*, an extension of the Shapley value introduced by Owen (1977) (see also Owen, 2013). The Owen value is characterized uniquely by the following axioms: efficiency, the null player property, symmetry within unions, symmetry in the quotient game, and additivity.

Carreras and Puente (2006) (see also Carreras & Puente, 2011) extended the binomial semivalues to games with a coalition structure: they used these values in the quotient game and the Shapley value within unions and obtained the *symmetric coalitional binomial semivalues*, a family depending on one parameter $q \in [0, 1]$ that includes (when $q = 1/2$) the *symmetric coalitional Banzhaf value* introduced by Alonso and Fiestras (2002). The only axiomatic difference between these new coalitional values and the Owen value is that the former satisfy the total power property whereas the latter satisfies efficiency.

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[1] Incidentally, we point out that the "Banzhaf α-indices", introduced by Carreras (2004) when generalizing the decisiveness notion studied in Carreras (2005), are multinomial probabilistic values applied to simple games—i.e., used as power indices. Cf. Carreras and Puente (2013) or Carreras and Puente (2014a) for a joint work on these values and Giménez, Llongueras, and Puente (2014) for their application to the study of the partnership formation, which generalizes Carreras, Llongueras, and Puente (2009) widely.

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In the present paper we extend the multinomial probabilistic values to games with a coalition structure by introducing the *coalitional multinomial probabilistic values*, a family that includes all symmetric coalitional binomial semivalues.

A first main aspect of the coalitional multinomial probabilistic values is that each one of them depends on n parameters (n being the number of players), which are interpreted as the individual tendencies of the players to form coalitions. A second main aspect is that they apply a multinomial probabilistic value in the quotient game that arises once the coalition structure is actually formed, but share within each union the payoff so obtained by applying the Shapley value to a game that concerns only the players of that union.

Using the Shapley value looks highly interesting in voting contexts. Indeed, once an alliance is formed—and, especially, if it supports a coalition government,—cabinet ministries, parliamentary and institutional positions, budget management, and other political responsibilities should be distributed among the members of the alliance *efficiently*, so the Shapley value is useful here. We thus evaluate not only the parliamentary power of the alliance but also the way to share this power allocation among its members. This two-step procedure (first power, then cake) offers a balanced approach for dealing with coalitional bargaining.

We emphasize the role of these new values as a consistent alternative to classical coalitional values. The fact that they are based on tendency profiles provides new tools to encompass a wide variety of situations that derive from players' personalities when playing a game. In this sense, the coalitional multinomial probabilistic values constitute a significant generalization of the symmetric coalitional binomial semivalues, whose monoparametric condition implies a limited capability to analyze such situations. Of course, these situations cannot be analyzed, without modifying the game, by means of classical, nonparametric values, which can be concerned only with the (formal) structure of the game.

Finally, it is worth mentioning that the greater ability of the new values to deal with strategic features is achieved without losing standard properties satisfied by classical coalitional values, e.g. linearity, positivity, the total power property, the dummy player property, symmetry within unions, or the quotient game property.

The organization of the paper is as follows. In Section 2, a minimum of preliminaries is provided. Section 3 is devoted to define and to study the family of coalitional multinomial probabilistic values, including an axiomatic characterization. In Section 4 we restrict such values to simple games. Section 5 presents a computation procedure for these values by means of multilinear extensions. Section 6 contains an application of the coalitional multinomial probabilistic values to the analysis of the Catalonia Parliament (Legislature 2003–2007). Section 7 includes concluding remarks. All proofs are collected in Appendix A.

2. Preliminaries

We assume that the reader is generally familiar with the basic ideas of the cooperative game theory (including simple games, which will be briefly revised in Section 4).

2.1. Games and values

Let N be a finite set of *players*, usually denoted as $N = \{1, 2, \ldots, n\}$, and 2^N be the set of *coalitions* (subsets of N). A *(cooperative) game* in N is a function $v : 2^N \to \mathbb{R}$ that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \le v(T)$ whenever $S \subset T \subseteq N$. Player $i \in N$ is a *dummy in* v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N\backslash\{i\}$, and *null in* v if, moreover, $v(\{i\}) = 0$. Two players $i, j \in N$ are *symmetric in* v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N\backslash\{i, j\}$.

Endowed with the natural operations for real-valued functions, i.e. $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set $\mathcal{G}_N$ of all games in N becomes a vector space. For every nonempty coalition $T \subseteq N$, the *unanimity game* u_T in N is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$.

By a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \to \mathbb{R}^N$, that assigns to every game v a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

The *multilinear extension* Owen (1972) of a game $v \in \mathcal{G}_N$ is the real-valued function defined on $\mathbb{R}^N$ by[2]

$$f(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq N} \left[\prod_{i \in S} x_i \prod_{j \in N\backslash S} (1 - x_j) \right] v(S). \tag{1}$$

2.2. Multinomial probabilistic values

The multinomial probabilistic values form a subfamily of *probabilistic values* Weber (1988). They were introduced in reliability by Puente (2000) (see also Freixas & Puente, 2002) as follows. Let $N = \{1, 2, \ldots, n\}$ and let $\mathbf{p} \in [0, 1]^n$, that is, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ with $0 \le p_i \le 1$ for $i = 1, 2, \ldots, n$, be given. Then the coefficients

$$p_S^i = \prod_{j \in S} p_j \prod_{\substack{k \in N\backslash S \\ k \neq i}} (1 - p_k) \quad \text{for all } i \in N \text{ and } S \subseteq N\backslash\{i\} \tag{2}$$

(where the empty product, arising if $S = \emptyset$ or $S = N\backslash\{i\}$, is taken to be 1) define a probabilistic value on $\mathcal{G}_N$ that is called the $\mathbf{p}$-*multinomial probabilistic value* and will be denoted here as $\lambda^{\mathbf{p}}$. Its action is then given by

$$\lambda_i^{\mathbf{p}}[v] = \sum_{S \subseteq N\backslash\{i\}} \left[\prod_{j \in S} p_j \prod_{\substack{k \in N\backslash S \\ k \neq i}} (1 - p_k) \right] [v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N. \tag{3}$$

In particular, the action of $\lambda^{\mathbf{p}}$ on a unanimity game u_T is given by

$$\lambda_i^{\mathbf{p}}[u_T] = \prod_{\substack{j \in T \\ j \neq i}} p_j \quad \text{if } i \in T \quad \text{and} \quad \lambda_i^{\mathbf{p}}[u_T] = 0 \quad \text{otherwise.} \tag{4}$$

As was announced in Section 1, we will attach to each p_i the meaning of *tendency of player i to form coalitions*, and thus we will say that $\mathbf{p}$ is a *(tendency) profile* in N. The components of $\mathbf{p}$ will be assumed to be independent of each other. From Eq. (2) it follows that coefficient p_S^i, the probability of i to join S according to Weber (1988), is an increasing function of the tendency of each member of S to form coalitions and a decreasing function of the tendency in this sense of each outside player, i.e. each member of $N\backslash(S \cup \{i\})$.

2.3. Games with a coalition structure

Given $N = \{1, 2, \ldots, n\}$, we will denote by $B(N)$ the set of all partitions of N. Each $B \in B(N)$ is called a *coalition structure* in N, and each member of B is called a *union*. The so-called *trivial coalition structures* are $B^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ (individual coalitions) and $B^N = \{N\}$ (grand coalition). A *(cooperative) game with a coalition structure* is a pair $[v; B]$, where $v \in \mathcal{G}_N$ and $B \in B(N)$ for a given N. Each partition B gives a pattern of cooperation among players. We denote by $\mathcal{G}_N^{cs} = \mathcal{G}_N \times B(N)$ the set of all games with a coalition structure and player set N.

If $[v; B] \in \mathcal{G}_N^{cs}$ and $B = \{B_1, B_2, \ldots, B_m\}$, the *quotient game* v^B is the game played by the unions or, rather, by the *quotient set* $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^B(R) = v\left(\bigcup_{r \in R} B_r\right) \quad \text{for all } R \subseteq M.$$

[2] The term "multilinear" means that, for each $i \in N$, the function is linear in x_i, that is, of the form $f(x_1, x_2, \ldots, x_n) = g_i(x_1, x_2, \ldots, \hat{x}_i, \ldots, x_n)x_i + h_i(x_1, x_2, \ldots, \hat{x}_i, \ldots, x_n)$.