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Stronger multi-commodity flow formulations of the Capacitated Vehicle Routing Problem

Adam N. Letchford [a], Juan-José Salazar-González [b,*]

[a] Department of Management Science, Lancaster University, Lancaster LA1 4YW, UK
[b] Department of Mathematics, Universidad de La Laguna, 38271 Tenerife, Spain

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ABSTRACT

The *Capacitated Vehicle Routing Problem* is a much-studied (and strongly $\mathcal{NP}$-hard) combinatorial optimization problem, for which many integer programming formulations have been proposed. We present two new *multi-commodity flow* (MCF) formulations, and show that they dominate all of the existing ones, in the sense that their continuous relaxations yield stronger lower bounds. Moreover, we show that the relaxations can be strengthened, in pseudo-polynomial time, in such a way that all of the so-called *knapsack large multistar* (KLM) inequalities are satisfied. The only other relaxation known to satisfy the KLM inequalities, based on set partitioning, is strongly $\mathcal{NP}$-hard to solve. Computational results demonstrate that the new MCF relaxations are significantly stronger than the previously known ones.

1. Introduction

Vehicle Routing Problems (VRPs) are classic problems in operational research and logistics, and have also received a great deal of attention from the combinatorial optimization community. A huge number of papers have been written on the theory and applications of VRPs, and on exact and heuristic solution methods for them (see, e.g., the edited volumes Ball, Magnanti, Monma, & Nemhauser, 1995; Golden, Raghavan, & Wasil, 2008; Toth & Vigo, 2001.)

This paper is concerned with the *Capacitated* VRP (CVRP), which Dantzig and Ramser (1959) defined as follows. A fleet of identical vehicles, with limited capacity, is located at a depot. There are n customers that require service. Each customer has a known demand. The cost of travel between any pair of customers, or between any customer and the depot, is also known. The task is to find a minimum-cost collection of vehicle routes, each starting and ending at the depot, such that each customer is visited by exactly one vehicle, and no vehicle visits a set of customers whose total demand exceeds the vehicle capacity.

Letchford and Salazar-González (2006) surveyed and compared several integer programming formulations of the CVRP. These included the so-called *two-* and *three-index* formulations, the *single-*, *two-* and *multi-commodity flow* formulations, and the *set partitioning* formulations. At present, the most successful exact algorithms for the CVRP are based on the two-index formulation (e.g., Lysgaard, Letchford, & Eglese, 2004) or on set partitioning formulations (e.g., Fukasawa et al., 2006, Baldacci, Christofides, and Mingozzi (2008)).

One way to measure the strength of an alternative formulation is to project the feasible region of its continuous relaxation into the space of the natural (two-index) formulation. Gouveia (1995) showed that, in the case of the single-commodity flow formulation, the projection satisfies a family of valid inequalities now known as *generalized large multistar* (GLM) inequalities. Letchford and Salazar-González (2006) showed that the projection of the set partitioning formulation (with only elementary routes permitted) satisfies the so-called *knapsack large multistar* (KLM) inequalities, defined by Letchford, Eglese, and Lysgaard (2002). The KLM inequalities include the GLM inequalities and the so-called *subtour elimination* (SE) inequalities as special cases. Unfortunately, the continuous relaxation of the set partitioning formulation is itself strongly $\mathcal{NP}$-hard to solve.

This paper has four main contributions. First, we show how to strengthen the two best multi-commodity flow (MCF) formulations, by adding only a polynomial number of additional constraints. Second, we show that the projections of our two formulations satisfy the GLM and SE inequalities. Third, we show that the new formulations can be further strengthened, in pseudo-polynomial time, in such a way that all of the KLM inequalities are satisfied. (We remark that no polynomial or pseudo-polynomial time separation algorithm is known for the KLM inequalities themselves.) Finally, we present some computational results that demonstrate that the new MCF formulations are significantly stronger than the previously known ones.

As mentioned above, the current best algorithms for the CVRP are based on two-index or set partitioning formulations. The contribution of this paper may therefore appear to be only of theoretical

* Corresponding author. Tel.: +34 922 318184; fax: +34 922 318170.
E-mail addresses: a.n.letchford@lancaster.ac.uk (A. N. Letchford), jjsalaza@ull.es (J.-J. Salazar-González).

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interest. We would like to point out, however, that there exist variants of the CVRP for which it is natural, or even essential, to use additional commodity-flow variables. This includes, for example, the problem described by Hernández-Pérez and Salazar-González (2009), in which several distinct products have to be picked up and delivered at various locations, and the one described by Kara, Kara, and Yetis (2007), in which the cost of traversing an arc is an increasing function of vehicle load. Potentially, our results could be used to derive better formulations and algorithms for such problems.

The structure of the paper is as follows. The literature is reviewed in Section 2. The strengthened MCF formulations are presented and analysed in Section 3. The result on KLM inequalities is given in Section 4. Some computational results are given in Section 5, and some concluding remarks are made in Section 6.

Throughout the paper, we use the following notation. We have a complete directed graph G with node set $V = \{0, 1, \ldots, n\}$ and arc set A. Node 0 represents the depot, and nodes $1, \ldots, n$ represent customers. We sometimes write V_c for $V \setminus \{0\}$, the set of customer nodes. The (positive integer) demand of customer $i \in V_c$ is q_i. The (positive integer) vehicle capacity is Q. The (non-negative integer) cost of traversing arc $(i, j) \in A$ is c_{ij}. (Our approach can easily be adapted to the case of symmetric costs and/or the case in which the number of vehicles is restricted.)

2. Literature review

As mentioned above, many formulations have been proposed for the CVRP. For brevity, we review only ones of relevance here. Subsections 2.1–2.4 cover *two-index vehicle flow*, *single-* and *two-commodity flow*, *multi-commodity flow* and *set partitioning* formulations, respectively.

2.1. The two-index vehicle flow formulation

Laporte and Nobert (1983) presented what is now called the *two-index vehicle flow* formulation. For all $(i, j) \in A$, define a binary variable x_{ij}, taking the value 1 if and only if some vehicle travels from i to j. For any $S \subset V$, let $\delta^+(S)$ (respectively, $\delta^-(S)$) denote the set of arcs (i, j) with $i \in S, j \in V \setminus S$ (respectively, with $i \in V \setminus S, j \in S$). If $S = \{i\}$ then we will write $\delta^+(i)$ and $\delta^-(i)$ rather than $\delta^+(\{i\})$ and $\delta^-(\{i\})$, for brevity. Given some $F \subset A$, let $x(F)$ denote $\sum_{(i,j)\in F} x_{ij}$. Finally, for any set of customers $S \subset V_c$, let $q(S) = \sum_{i\in S} q_i$. Then the formulation is:

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \qquad (1)$$

$$\text{s.t.} \quad x(\delta^+(i)) = 1 \qquad (i \in V_c) \qquad (2)$$

$$x(\delta^-(i)) = 1 \qquad (i \in V_c) \qquad (3)$$

$$x(\delta^+(S)) \geq \lceil q(S)/Q \rceil \qquad (S \subseteq V_c) \qquad (4)$$

$$x_{ij} \in \{0, 1\} \qquad ((i, j) \in A). \qquad (5)$$

The *out-degree equations* (2) and the *in-degree equations* (3) ensure that vertices are visited exactly once. The constraints (4), called *rounded capacity* (RC) inequalities, prevent the existence of infeasible routes, and also have the side-effect of preventing subtours. Finally, (5) are the integrality conditions on the x-variables.

Several families of valid linear inequalities (cutting planes) have been developed for the two-index vehicle flow formulation (see Naddef & Rinaldi, 2001 for a survey). We will be interested in the following inequalities:

- The *fractional capacity* (FC) inequalities:

$$x(\delta^+(S)) \geq \frac{q(S)}{Q} \qquad (S \subseteq V_c). \qquad (6)$$

- The *subtour elimination* (SE) inequalities:

$$x(\delta^+(S)) \geq 1 \qquad (S \subseteq V_c). \qquad (7)$$

- The *generalized large multistar* (GLM) inequalities (see Gouveia, 1995):

$$x(\delta^+(S)) \geq \frac{1}{Q} \sum_{i\in S} \left(q_i + \sum_{j\in V_c\setminus S} q_j(x_{ij} + x_{ji}) \right) \qquad (S \subseteq V_c). \qquad (8)$$

- The *knapsack large multistar* (KLM) inequalities (see Letchford et al., 2002):

$$x(\delta^+(S)) \geq \frac{1}{\beta} \sum_{i\in S} \left(\alpha_i + \sum_{j\in V_c\setminus S} \alpha_j(x_{ij} + x_{ji}) \right) \qquad (S \subseteq V_c), \qquad (9)$$

where $\alpha \geq 0$ and $\beta > 0$ are such that the inequality $\sum_{i\in V_c} \alpha_i y_i \leq \beta$ is valid for the 0-1 knapsack polytope:

$$\text{KP}(Q, q) := \text{conv} \left\{ y \in \{0, 1\}^n : \sum_{i\in V_c} q_i y_i \leq Q \right\}. \qquad (10)$$

Obviously, the RC inequalities dominate the FC and SE inequalities, and the GLM inequalities dominate the FC inequalities. It is also not difficult to see that the KLM inequalities include the GLM and SE inequalities as special cases. In general, there are no other dominance relations.

2.2. Single- and two-commodity flow formulations

The first single-commodity flow formulation, that we call "SCF1", was presented by Gavish and Graves (1979). A continuous variable f_{ij} is defined for each $(i, j) \in A$, representing the total load (if any) carried along the arc (i, j). One then replaces constraints (4) in the two-index vehicle flow formulation with:

$$f(\delta^-(i)) - f(\delta^+(i)) = q_i \qquad (i \in V_c) \qquad (11)$$

$$0 \leq f_{ij} \leq Qx_{ij} \qquad ((i, j) \in A). \qquad (12)$$

The constraints (11) ensure that each customer i receives the demand of q_i. The constraints (12) are just bounds on the f variables.

Gavish (1984) proposed to strengthen SCF1 by replacing the bounds (12) with the stronger bounds

$$q_j x_{ij} \leq f_{ij} \leq (Q - q_i)x_{ij} \qquad ((i, j) \in A).$$

We call this strengthened formulation "SCF2".

Gouveia (1995) used Hoffman's circulation theorem (Hoffman, 1960) to project the feasible regions of the LP relaxations of SCF1 and SCF2 into the space of the x variables. The projection of SCF1 is given by the out-degree equations (2), the in-degree equations (3), the FC inequalities (6) and non-negativity. The projection of SCF2, as expected, is stronger, satisfying the GLM inequalities (8) in place of the FC inequalities.

In the paper of Baldacci, Hadjiconstantinou, and Mingozzi (2004), a two-commodity flow formulation is presented for the case in which the costs c_{ij} are symmetric. Letchford and Salazar-González (2006) show that, in this case, the LP relaxation of their formulation gives the same lower bound as that of SCF2.

2.3. Multi-commodity flow formulations

The first multi-commodity flow formulation, that we call "MCF1a", was presented by Garvin, Crandall, John, and Spellman (1957). A binary variable f^k_{ij} is defined for each $k \in V_c$ and each $(i, j) \in A$, taking the value 1 if and only if a vehicle traverses (i, j) on the way from the depot to k. The formulation is then obtained by replacing constraints (4) in the two-index vehicle flow formulation with:

$$f^k(\delta^+(0)) = f^k(\delta^-(k)) = 1 \qquad (k \in V_c) \qquad (13)$$

$$f^k(\delta^-(0)) = f^k(\delta^+(k)) = 0 \qquad (k \in V_c) \qquad (14)$$

$$f^k(\delta^+(i)) = f^k(\delta^-(i)) \qquad (k, i \in V_c : i \neq k) \qquad (15)$$