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Discrete Optimization

A continuous time–cost tradeoff problem with multiple milestones and completely ordered jobs

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ABSTRACT

We consider a continuous time–cost tradeoff problem with multiple milestones and completely ordered jobs. If a milestone is tardy, a penalty cost may be imposed. The processing times of jobs can be compressed by additional resources or activities that incur compression costs. The objective is to minimize the total penalty cost plus the total compression cost. We show that the problem is NP-hard, even if the compression cost is described as a concave function, and we present a pseudo-polynomial time algorithm for that case. Furthermore, we show that the problem is polynomially solvable if the compression cost function is convex.

1. Introduction

The time–cost tradeoff problem (TCTP) assumes that each job's processing time can be compressed by using additional resources such as equipment, labor, or capital (Ford & Fulkerson, 1962; Kelley, 1961). The typical objective is to minimize the project completion time subject to a constraint on the budget for compression or to minimize the total cost for compression subject to a constraint on the project completion time.

The *continuous* TCTP (CTCTP) is defined as a TCTP in which the compression cost is described as a continuous function. The *linear* TCTP (LTCTP) is a special case of the CTCTP such that the compression cost function is linear. The LTCTP can be formulated as a linear programming problem and efficiently solved by the network flow method introduced by Fulkerson (1961) and Kelley (1961). For a comprehensive review of more general models (e.g., resource-constrained project scheduling), see Artigues, Demassey, and Neron (2008); Brucker, Drexl, Möhring, Neumann, and Pesch (1999); Demeulemeester and Herroelen (2002); Weglarz (1999); Weglarz, Jozefowska, Mika, and Waligora (2011).

The TCTP studies above assume a single milestone for the overall project, that is, the last job. In reality, however, multiple milestones can exist within a project. Thus, if an appointed task is not completed at the milestone, a penalty is imposed. For example, a venture capital company makes small investments in a project at first and then, when a milestone has been reached, determines whether to stop the project or make more investments, depending on the progress of the project (Bell, 2000; Sahlman, 1994). To the best of our knowledge, no study has been conducted on a TCTP with more than one milestone, except that of Choi and Chung (2014). These authors considered two LTCTPs with multiple milestones and completely ordered jobs. The objective of the first problem is to minimize the total weighted number of tardy jobs subject to a constraint on the total compression cost. The second objective is to minimize the total weighted number of tardy jobs plus the total compression cost. The authors proved the NP-hardness of the first problem and the polynomiality of the second.

This paper considers a generalized version of the second problem introduced by Choi and Chung (2014), in which the compression cost function can be nonlinear. The time–cost relation has been studied in several papers. Nonlinear (Moussourakis & Haksever, 2010), concave (Falk & Horowitz, 1972), and convex compression cost functions (Berman, 1964; Lamberson & Hocking, 1970) have been studied for the TCTP.

Our problems can be formally stated as follows. The CTCTP is represented by a directed *activity-on-node* graph $G = (V, A)$, where $V = \{1, 2, \ldots, n\}$ is the set of jobs and A is the set of precedence relations. Relation $(i, j) \in A$ means that job i should be completed before job j is started. Associated with job j is a normal processing time $p_j \in \mathbb{Z}_+$ and a maximal compression amount $u_j (\le p_j) \in \mathbb{Z}_+$, $j = 1, 2, \ldots, n$, where $\mathbb{Z}_+$ is the set of nonnegative integers. Let $x = (x_1, x_2, \ldots, x_n)$ be a real vector for which x_j is the compressed amount of job j and $0 \le x_j \le u_j$, $j = 1, 2, \ldots, n$. When job j is compressed by x_j, the compression cost becomes $f_j(x_j)$, where $f_j(x_j)(\ge 0)$

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is a non-decreasing continuous function of x_j. Throughout the paper, without loss of generality, we consider only

- Schedules in which each job is processed as soon as possible by not allowing unnecessary idle time;
- Compression cost functions that are either concave or convex.

The concavity of the compression cost function means increasing marginal returns, such that, as we input more resources, the addition of resources become more productive. In other words, two workers are more than twice as productive as one worker and four workers are more than twice as productive as two workers. The convexity reflects the general law of diminishing marginal returns, such that, as the number of new workers increases, the marginal productivity of an additional worker is less than that of the previous workers. Note that the relationship between the investment of the additional resources and the marginal productivity is corresponding to the one between the compression cost function and the compression amount in our model.

Since each job has a unique starting time under each value of x, let x be termed the schedule. Let $D \subseteq V$ be the set of milestones. Note that each milestone corresponds to a specific job. For $j \in D$, let $w_j \in \mathbb{Z}_+$ and $d_j \in \mathbb{Z}_+$ be the penalty cost for tardiness and the due date of milestone j, respectively. Let $C_j(x)$ be the completion time of job j under x. Then, our problem is defined as

$$\begin{aligned} \text{minimize} \quad & \sum_{j\in T(x)} w_j + \sum_{j=1}^{n} f_j(x_j) \\ \text{subject to} \quad & C_i(x) + p_j - x_j \le C_j(x), \quad \forall (i,j) \in A, \\ & 0 \le x_j \le u_j, \quad j = 1, 2, \ldots, n, \end{aligned}$$

where $T(x) = \{j \in D \mid C_j(x) > d_j\}$ is the set of tardy milestones under x.

We can transform the problem with $D \subset V$ into one with $D = V$ by letting $d_i = \sum_{j=1}^{n} p_j$ for $i \in V \setminus D$. It is clear that the optimal schedules for these two problems are identical, since job i for $i \in V \setminus D$ will not be completed late. Thus, for simplicity of notation, we assume that $D = V$.

In this paper, we focus on the CTCTP with multiple milestones and completely ordered jobs. Completely ordered jobs can be described by a chain graph

$$A = \{(1,2), (2,3), \ldots, (n-1, n)\}.$$

Hence, let our problem be referred to as *CTCTP-chain*.

Then, the CTCTP-chain can be formulated as

$$\begin{aligned} \text{minimize} \quad & \sum_{j\in T(x)} w_j + \sum_{j=1}^{n} f_j(x_j) \\ \text{subject to} \quad & 0 \le x_j \le u_j, \ j = 1, 2, \ldots, n, \end{aligned}$$

where $T(x) = \{j \in D \mid \sum_{i=1}^{j} (p_i - x_i) > d_j\}$ is the set of tardy milestones under x. The CTCTP-chain is motivated by a product development process following a sequential pattern in which information about the product tends to be accumulated slowly in consecutive stages (Roemer & Ahmadi, 2004). Each stage can begin only when its immediately preceding stage obtains complete and final information after termination.

The remainder of the paper is organized as follows. Section 2 introduces properties of some optimal schedules for the CTCTP-chain. Section 3 shows that the CTCTP-chain with a concave compression cost function, referred to as a *concave-CTCTP-chain*, is NP-hard and can be solved in pseudo-polynomial time. Section 4 shows that the CTCTP-chain with a convex compression cost function, referred to as a *convex-CTCTP-chain*, is polynomially solvable. The final section presents concluding remarks and discusses future work.

2. Properties of some optimal schedules for the CTCTP-chain

In this section, we introduce properties of some optimal schedules for the CTCTP-chain. Let job j be referred to as a just-in-time (JIT) job in x if it is completed exactly on its due date, that is, $d_j = C_j(x)$.

Lemma 1. *There exists an optimal schedule satisfying one of the following conditions:*

(i) All jobs are uncompressed.
(ii) There exists at least one JIT job.

Proof. Let x^* be an optimal schedule with the most JIT jobs. If x^* satisfies (i) or (ii), then the proof is complete. Now, suppose that an optimal schedule x^* has compressed jobs and no JIT job. Let job l be the last compressed job in x^* and let E be the set of non-tardy jobs in $\{l, \ldots, n\}$. Since f_l is non-decreasing, we can construct a new optimal schedule $\bar{x}$ by letting $\bar{x}_l = x^*_l - \min\{x^*_l, \Delta\}$ and $\bar{x}_j = x^*_j$ for $j \in V \setminus \{l\}$, where $\Delta = \min\{d_j - C_j(x^*) \mid j \in E\}$. If $\Delta \neq x^*_l$, then at least one job in E becomes a JIT job, which is a contradiction of the definition of x^*. If $\Delta = x^*_l$, then job l is uncompressed. By repeatedly applying this argument, we can construct an optimal schedule satisfying (i) or (ii). □

By Lemma 1, an optimal schedule can be found by comparing a schedule with all uncompressed jobs and an schedule with minimum costs among schedules with at least one JIT job. Henceforth, we consider only schedules in which the number of JIT jobs is at least one.

Lemma 2. *There exists an optimal schedule such that jobs processed after the last JIT job are uncompressed.*

Proof. Since a similar argument in the proof of Lemma 1 holds, we omit the details. □

Henceforth, we only consider those schedules satisfying Lemmas 1 and 2.

3. Concave-CTCTP-chain

In this section, we show that the concave-CTCTP-chain is NP-hard and introduce an approach to solve it in pseudo-polynomial time.

3.1. NP-hardness

In this section, we show that the decision version of the concave-CTCTP-chain is NP-complete by reduction from the partition problem, which is known to be NP-complete (Garey & Johnson, 1979).

The partition problem can be stated as follows: Given m positive integers $a_1, a_2, \ldots, a_m$ whose sum is even, is there a subset $I \subset \{1, 2, \ldots, m\}$ such that $\sum_{j\in I} a_j = \frac{1}{2}\sum_{j=1}^{m} a_j$?

Theorem 1. *The decision version of the concave-CTCTP-chain is NP-complete.*

Proof. The decision version of the concave-CTCTP-chain is stated as follows: Given a threshold K, find a schedule x such that

$$\sum_{j\in T(x)} w_j + \sum_{j=1}^{n} f_j(x_j) \le K.$$

It is clear that the decision version of the concave-CTCTP-chain is in NP. Henceforth, we reduce the partition problem to the decision version of the concave-CTCTP-chain. Given an instance of the partition problem, we construct an instance of the concave-CTCTP-chain asfollows. Let $n = m$. Given n jobs such that for $j = 1, 2, \ldots, n$, let