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Decision Support

A cyclical square-root model for the term structure of interest rates

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A B S T R A C T

This paper presents a cyclical square-root model for the term structure of interest rates assuming that the spot rate converges to a certain time-dependent long-term level. This model incorporates the fact that the interest rate volatility depends on the interest rate level and specifies the mean reversion level and the interest rate volatility using harmonic oscillators. In this way, we incorporate a good deal of flexibility and provide a high analytical tractability. Under these assumptions, we compute closed-form expressions for the values of different fixed income and interest rate derivatives. Finally, we analyze the empirical performance of the cyclical model versus that proposed in Cox et al. (1985) and show that it outperforms this benchmark, providing a better fitting to market data.

1. Introduction

Through the time, modeling the term structure of interest rates (TSIR) has been the object of many studies and the aim of attention for economists and financial institutions. This paper proposes a cyclical square-root model where the instantaneous interest rate is pulled back to a certain time-dependent long term level characterized by an harmonic oscillator. Therefore, assuming a time-dependent mean reversion level will derive in a time-dependent spot rate volatility. Empirical evidence (see, for instance, Amin & Morton (1994) Chan, Karolyi, Longstaff, & Sanders (1992)) illustrated that interest rate volatility depends on the interest rates level. Then, it seems natural to model interest rate volatility using a similar functional form as that in the mean reversion level.

Models proposed in the academic literature can be classified in endogenous and exogenous. Endogenous models make certain assumptions on the factors that drive the TSIR and their stochastic processes. The TSIR is fully characterized by these factors meaning that the current TSIR is an output rather than an input of the model. Examples of one-factor models are Brennan and Schwartz (1980), Cox, Ingersoll, and Ross (1985), or Vasicek (1977) (CIR from now on). The main drawback of these models is the lack of empirical realism as they do not fit accurately the current TSIR and, consequently, do not price correctly fixed income assets. In order to cope with this problem, we can find two-factor models such as, for instance, Cox et al. (1985), Longstaff and Schwartz (1992), or Schaefer and Schwartz (1984) while Babbs and Nowman (1999), Balduzzi, Das, Foresi, and Sundaram (1996), Beaglehole and Tenny (1991), Chen (1996), Dai and Singleton (2000), and Duffie and Kan (1996) introduced and analyzed different multi-factor models.

On the other hand, exogenous models consider the current TSIR as an input and derive future changes in interest rates avoiding intertemporal arbitrage opportunities. The first contribution was made by Ho and Lee (1986) who showed how to build a model consistent with the initial TSIR. Since this model has some disadvantages, their work has been extended by a number of authors such as Abaffy, Bertocchi, and Gnudi (2005), Black, Derman, and Toy (1990), Black and Karasinski (1991), Brigo, Mercurio, and Morini (2005), Heath, Jarrow, and Morton (1992), Hull and White (1990, 1993), and Mercurio and Moraleda (2000).

Applications and analysis of some of these models can be found in Chen and Huang (2013), Chen and Hu (2010), Chiarella, Clewlow, and Musti (2005), Chiarella, Fanelli, and Musti (2011), Date and Wang (2009), de Frutos (2008), Falini (2010), Hainaut (2009), Hernández, Saunders, and Seco (2012), Mitra, Date, Mamon, and Wang (2013), and Weissensteiner (2010), among others. Excellent literature reviews on term structure TSIR models can be seen in some books as, for instance, Andersen and Piterbarg (2010), Brigo and Mercurio (2006), Cairns (2004), Filipović (2009), Hunt and Kennedy (2004), James and Webber (2001), Munk (2011), Nawalkha, Believa, and Soto (2007), and Rebonato (1998) or papers as Boero and Torricelli (1996), Schmidt (2011), or Vetzal (1994), among others.

Derivative markets trade a huge volume of contracts on a daily basis and derivatives pricing has become an issue of utmost

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importance. Despite the great progress in this matter, there is still a trade-off between analytical tractability and empirical accuracy. In this paper, we introduce a model where the mean reversion level and the spot rate volatility follow a cyclical process characterized by an harmonic oscillator. This cyclical model provides great flexibility to reflect the different shapes that the TSIR can exhibit empirically and provides a high analytical tractability, allowing an accurate fitting of the TSIR and constituting a powerful pricing tool. Under this framework, we analytically price zero-coupon bonds and different derivatives such as forward on bonds, European options on zero-coupon and coupon-bearing bonds, European bond forward options, swaps, swaptions, caps, floors, collars, and provide some risk management measures. Finally, we analyze the empirical performance of this model versus its natural benchmark, the CIR model. We show that, for the data set used in this analysis, the cyclical model outperforms this benchmark, providing a much better fitting to current data for every time horizon.

This paper is organized as follows. Section 2 introduces the cyclical model and its practical implications. Section 3 presents the general pricing partial differential equation and derives closed-form expressions for different derivatives. Section 4 presents the empirical analysis of the model. Finally, Section 5 summarizes the main findings and conclusions. Mathematical proofs are deferred to Appendix A.

2. A cyclical square-root model for the term structure

In this section, we propose our model and the specific functional form for each time-dependent parameter, and describe all the practical implications arising from this model.

Let r_t denote the instantaneous interest rate available at time t whose dynamics is

$$dr_t = \mu_r\, dt + \sigma_r\, dW_t \tag{1}$$

where W_t is a standard Wiener process and

$$\mu_r = \kappa(\theta_t - r_t) \tag{2}$$

$$\sigma_r = \sigma_t\sqrt{r_t} \tag{3}$$

where $\kappa \in \mathbb{R}^+$.

Consider the harmonic oscillator given as $f(t) = A\sin(\varphi - \omega t)$, where A, φ, and ω denote the amplitude, offset phase, and temporal frequency, respectively. This function provides a simple and flexible functional form to represent a cyclical behavior. In addition, working with an harmonic oscillator instead of a high-order polynomial provides a good deal of analytical tractability.

Departing from this harmonic oscillator, we assume that the mean reversion level, θ_t, and the volatility, σ_t^2, in Eqs. (2) and (3), are defined as

$$\theta_t = A_\theta \sin^2(\varphi - \omega t) \tag{4}$$

$$\sigma_t^2 = A_\sigma \sin^2(\varphi - \omega t) \tag{5}$$

These specific expressions guarantee the positiveness of the mean reversion level and the interest rate volatility. It is clear that this model nests the CIR one taking $\omega = 0$ in Eqs. (4) and (5). Note that we incorporate two additional parameters, phase and frequency, with respect to the CIR model.

For square-root processes of this type, Cox et al. (1985) shows that the distribution function of interest rates is a rescaled non-central chi-square with δ degrees of freedom. Note that, whenever δ is not a positive integer, the distribution of r_t is unknown. Besides, the dimension of the process r_t is given by $\delta = \frac{4\theta_t\kappa}{\sigma_t^2}$. Eqs. (4) and (5) illustrate that both waves are in phase, then the model's dimension can be represented as $\delta = \frac{4A_\theta\kappa}{A_\sigma} > 0$.[2]

[2] Note that, if $\sin(\varphi - \omega t)$ is equal to zero, then δ becomes indeterminate. As this case would only occur for a infinitesimal period of time, we do not consider this possibility.

Our model guarantees the positiveness of interest rates. On this respect, Feller (1951) studied the Fokker–Plank–Kolmogorov equation for the transition density and showed that $r_t > 0$ if $\delta \geqslant 2$, however it can become null if $\delta < 2$ but will never become negative.

3. Pricing

This section presents closed-form expressions for the price of zero-coupon bonds and, later, we analytically compute closed-form formulas for the prices of different securities.

Let $P(r_t, t, T)$ denote the price at time t of a zero-coupon bond that pays \$1 at maturity T. Then, the bond price dynamics is given by the process

$$dP = \mu_P(r_t,t,T)P(r_t,t,T)dt + \sigma_P(r_t,t,T)P(r_t,t,T)dW_t \tag{6}$$

Applying Itô's Lemma and using (1), it can be shown that

$$\mu_P(r_t,t,T) = \frac{1}{P}\left(P_t + \mu_r P_r + \frac{1}{2}\sigma_r^2 P_{rr}\right) \tag{7}$$

$$\sigma_P(r_t,t,T) = \sigma_r \frac{P_r}{P} \tag{8}$$

where subscripts in P indicate the corresponding partial derivative. Applying standard no-arbitrage arguments, there exists a factor $\Lambda(r_t, t)$, called market price of risk, such that

$$\mu_P(r_t,t,T) - r_t = \Lambda(r_t,t)\sigma_P(r_t,t,T) \tag{9}$$

Finally, some trivial algebra leads to the following partial differential equation (PDE)

$$P_t(r_t,t,T) + (\mu_r - \Lambda(r_t,t)\sigma_r)P_r(r_t,t,T) + \frac{1}{2}\sigma_r^2 P_{rr}(r_t,t,T) - r_tP(r_t,t,T) = 0 \tag{10}$$

that must be verified by the price of any derivative.

3.1. Bond pricing

Similarly to Cox et al. (1985), we consider a market price of risk such as

$$\Lambda(r_t,t) = \frac{\lambda_t\sqrt{r_t}}{\sigma_t} \tag{11}$$

Using expressions (2), (3), and (11), the PDE (10) becomes

$$P_t(r_t,t,T) + (\kappa(\theta_t - r_t) - \lambda_t r_t)P_r(r_t,t,T) + \frac{1}{2}\sigma_t^2 r_t P_{rr}(r_t,t,T) - r_tP(r_t,t,T) = 0 \tag{12}$$

The solution of this PDE, subject to the boundary condition $P(r_T, T, T) = 1,\ \forall r_T$ is given by the following Proposition.

Proposition 1. *The price at time t of a zero-coupon bond with maturity T and \$1 face value is given by*

$$P(r_t,\tau) = A(\tau)e^{-B(\tau)r_t}$$

where

$$A(\tau) = \exp\left(-\int_t^T \kappa\theta_t B(\tau)dt\right)$$

$$B(\tau) = \frac{c_1MC(a,q,x) + MS(a,q,x)}{\frac{1}{2}(\lambda+\kappa)(c_1MC(a,q,x)+MS(a,q,x)) + \omega(c_1MCP(a,q,x)+MSP(a,q,x))}$$

$$a = -\frac{A_\sigma + (\lambda+\kappa)^2}{4\omega^2}$$

$$q = -\frac{A_\sigma}{8\omega^2}$$

$$x = \varphi - \omega t$$

$$c_1 = -\frac{MS(a,q,\varphi-\omega T)}{MC(a,q,\varphi-\omega T)}$$

$$\tau = T - t$$

where θ_t is given by (4), MC and MS represent the Mathieu cosine and sine function, respectively, and MCP and MSP represent the derivative