European Journal of Operational Research 243 (2015) 248–257

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Values of games with weighted graphs

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ARTICLE INFO

Article history:
Received 10 February 2014
Accepted 19 November 2014
Available online 29 November 2014

Keywords:
TU-game
Weighted graph
Myerson value
Fairness
Balanced contributions

ABSTRACT

In this paper we deal with TU games in which cooperation is restricted by means of a weighted network. We admit several interpretations for the weight of a link: capacity of the communication channel, flow across it, intimacy or intensity in the relation, distance between both incident nodes/players, cost of building or maintaining the communication link or even probability of the relation (as in Calvo, Lasaga, and van den Noweland, 1999). Then, according to the different interpretations, we introduce several point solutions for these restricted games in a way parallel to the familiar environment of Myerson. Finally, we characterize these values in terms of the (adapted) component efficiency, fairness and balanced contributions properties and we analyze the extent to which they satisfy a link/weight monotonicity property.

1. Introduction

A cooperative TU game describes a situation in which several actors can obtain certain transferable payoffs by means of the cooperation. Mathematically a TU game consists of a set of players and a characteristic function that assigns to each subset of players (coalition) a real number representing the worth of the coalition.

In this paper we deal with TU games in which cooperation is restricted by means of a weighted network. The seminal work on games in which restrictions in the cooperation are given by a graph is due to Myerson (1977). He assumed that the nodes in the graph are the players in the game, each link representing a direct bilateral communication channel. From this starting point, he defined the graph-restricted game and proposed as a point solution for players in this environ the so called Myerson value, i.e., the Shapley value of the restricted game. Moreover he characterized this value in terms of component efficiency and fairness. Later, Myerson (1980) gave another characterization of his value replacing fairness by balanced contributions.

In the Myerson model, bilateral restrictions in communication are dichotomous: they exist or they do not. Nevertheless, as has long been appreciated, each direct connection can be only partially (not totally) limited. We use in this paper a weighted graph as a model of these partial restrictions in the communications. A weighted graph consists of a set of nodes and a set of links, each link having an associated weight, a non-negative real number that can be interpreted in different ways: the capacity or the capability of the communication channel, the flow across it, the degree of intimacy, intensity or frequency if the link represents a social relation, the distance between both incident nodes, or even the cost of building or maintaining the communication link. Calvo, Lasaga, and van den Noweland (1999) introduced a probabilistic model in which the weight of a link is the probability of establishing the relation, these probabilities being independent. Later, Gómez, González-Arangüena, Manuel, and Owen (2008) generalized this model omitting the independence hypothesis. In a related context, Jiménez-Losada, Fernández, Ordoñez, and Grabisch (2010) consider cooperative TU games with Choquet players in which the restrictions in the communications are modeled using an (undirected) fuzzy graph.[1]

Other authors have considered values for games with node-weighted communication graphs. For example, in a game theoretical centrality analysis of terrorist networks, Lindelauf, Hamers, and Husslage (2013), Husslage, Borm, Burg, Hamers, and Lindelauf (2014) and Michalak et al. (2013) introduce a weighted connectivity game and propose its Shapley value as a centrality measure. This game can depend on information about relationship between terrorists in the network (link weights) and/or about individuals (node weights).

In this paper we introduce point solutions for games with restrictions in communications modeled by a weighted graph, following a parallel way to the familiar territory of Myerson.

In a TU game, each set of players is able to obtain its total dividend Harsanyi (1959). The basic idea underlying the definition of the Myerson game is that, under restrictions in the communications, every coalition that can be connected in the graph (possibly using intermediaries) obtains all its dividend in the game. Otherwise, i.e., in absence of connectedness, its dividend vanishes. And thus, the binary

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[1] For details on fuzzy graphs, readers can see Mordeson and Nair (2000).

http://dx.doi.org/10.1016/j.ejor.2014.11.033
0377-2217/

framework of communication is projected in an all or nothing possibility of obtaining the dividend. Nevertheless, the weighted graph introduces a non dichotomous but fuzzy scheme of relations. In accordance with this, we propose to consider a weighted graph game in which players partially connected obtain a part but not all their dividend and the lack of connectedness leading, of course, to the loss of the dividend. We introduce several forms of calculating this fraction of the dividend taking into account various alternative interpretations of the weights. Then, we use the Shapley value of the corresponding weighted graph restricted game as an appropriate solution concept for this situation and we characterize it in terms of the (adapted to this framework) component efficiency, fairness and balanced contributions properties. Finally, we explore the extent to which the defined values satisfy a link/weight monotonicity property.

The remainder of the paper is organized as follows. In Section 2 we introduce some notation and preliminaries; in Section 3 we define the weighted graph restricted games and the different extensions of the Myerson value in accordance with the various interpretations of the weights. Section 4 is devoted to characterization of the defined values. In Section 5 we deal with the link/weight monotonicity property and the paper ends with a section of final remarks and conclusions.

2. Preliminaries

A cooperative n-person game or a *TU-game* is a pair (N, v), $N = \{1, \dots, n\}$ being the set of players and $v : 2^N \to \mathbb{R}$, *the characteristic function,* a map satisfying $v(\emptyset) = 0$. For each coalition $S \subseteq N$, $v(S)$ represents the transferable utility that S can obtain whenever its members cooperate.

We will use G^N to denote the set of all TU-games with players set N. It is easy to see that G^N is a vector space. The game (N, u_S), $\emptyset \neq S \subseteq N$, with characteristic function given by:

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T \\ 0, & \text{otherwise} \end{cases}$$

is known as the *unanimity game* corresponding to S. The family of all games $\{(N, u_S)\}_{\emptyset \neq S \subseteq N}$ is a basis for G^N. As a consequence, the characteristic function v of every game in G^N can be written:

$$v = \sum_{\emptyset \neq S \subseteq N} \Delta_v(S) u_S.$$

The coefficients (coordinates) of v in such a basis are known as *Harsanyi dividends* (Harsanyi, 1959). The worth of every coalition S can be written in terms of its Harsanyi dividends. For each $S \subseteq N$, $S \neq \emptyset$:

$$v(S) = \sum_{\emptyset \neq T \subseteq S} \Delta_v(T).$$

A very popular point solution for TU-games is *the Shapley value* (Shapley, 1953), which assigns to every player the following convex linear combination of his marginal contributions to different coalitions:

$$\text{Sh}_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} (v(S \cup \{i\}) - v(S)), \quad i \in N.$$

An alternative expression for this value in terms of the dividends is:

$$\text{Sh}_i(N, v) = \sum_{i \in S \subseteq N} \frac{\Delta_v(S)}{s}, \quad i \in N.$$

A graph or a network is a pair (N, γ), $N = \{1, 2, \dots, n\}$ being the set of nodes and γ a subset of $\gamma_N = \{\{i, j\}, i, j \in N, i \neq j\}$. Each *link* $\{i, j\} \in \gamma$ represents a direct relation or a communication channel between i and j. Γ^N denotes the set of all graphs with nodes set N.

Given $(N, \gamma) \in \Gamma^N$, we will say that two nodes i and j are *directly connected in* γ if $\{i, j\} \in \gamma$. And we will say they are *connected* in γ if there exists a sequence of nodes $i_1, i_2, \dots, i_k$ with $i_1 = i$, $i_k = j$ such that $\{i_l, i_{l+1}\} \in \gamma$, for $l = 1, \dots, k-1$. A set $S \subseteq N$ is connected in γ if every pair of nodes in S is connected. *A connected component,* C, of the graph (N, γ) is a maximal connected subset. That is, C is connected and, for all $C' \subseteq N$, if $C \subsetneq C'$ then, C' is not connected. A graph (N, γ) induces a partition N/γ of the set N in connected components.

Given a set $S \subseteq N$ and a graph (N, γ), the restriction of the graph γ to the set S is the graph $(S, \gamma_{|S})$. Let S/γ be the set of the connected components of S in $(S, \gamma_{|S})$. A *subgraph* of a graph (N, γ) is a graph (N, γ') with $\gamma' \subseteq \gamma$.

Given a graph (N, γ) and a link $l \in \gamma$, $(N, \gamma \setminus \{l\})$ is the subgraph obtained when the relation l is severed and (N, γ_{-i}) is the resulting subgraph when all the incident in i links are broken and then i becomes an isolated node in the resulting graph.

Suppose that $(N, \gamma) \in \Gamma^N$ and let $(R, \gamma_{|R})$ denote the restriction to $R \subseteq N$, $R \neq \emptyset$. Suppose that $S \subseteq R$, $S \neq \emptyset$. If $S = \{i\}$ then the (unique) *connection subgraph of* S in $(R, \gamma_{|R})$ is defined as $(\{i\}, \emptyset)$. If $|S| > 1$, then a (not necessarily unique) *connection subgraph of* S in $(R, \gamma_{|R})$ is a graph $(D(\eta), \eta)$ satisfying:

(i) $\eta \subseteq \gamma_{|R}$ and S is connected in η, and
(ii) $D(\eta) = \{i \in N$ such that there exists $j \in N$ with $\{i, j\} \in \eta\}$.

A connection subgraph of $S \subseteq R$, $S \neq \emptyset$, in $(R, \gamma_{|R})$ is *minimal* [2] if there is no other connection subgraph $(D(\eta'), \eta')$ of S in $(R, \gamma_{|R})$ with $\eta' \subsetneq \eta$. Given a graph (N, γ) and $\emptyset \neq S \subseteq R \subseteq N$, $\mathcal{CG}(S, R, \gamma)$ will denote the family of all connection graphs of S in $(R, \gamma_{|R})$. Of course, it can occur that, for some S, R and (N, γ), $\mathcal{CG}(S, R, \gamma) = \emptyset$. Nevertheless, when we make use of it, this set will always be non empty. $\mathcal{MCG}(S, R, \gamma)$ will denote the family of all minimal connections graphs of $S \subseteq R$, $S \neq \emptyset$, in $(R, \gamma_{|R})$.

An *unweighted communication situation* or simply a *communication situation* is a triple (N, v, γ), (N, v) being a TU game and (N, γ) a graph. We will use $\mathcal{CS}^N$ to denote the set of all communication situations with players-nodes set N. An *allocation rule* ψ on $\mathcal{CS}^N$ is a map $\psi : \mathcal{CS}^N \to \mathbb{R}^n$, $\psi_i(N, v, \gamma)$ representing the outcome for player i in game (N, v) given the restrictions in the communication imposed by the graph (N, γ).

The *Myerson value* (Myerson, 1977) is the allocation rule μ on $\mathcal{CS}^N$ defined:

$$\mu(N, v, \gamma) = \text{Sh}(N, v^\gamma), \quad \text{where } v^\gamma(S) = \sum_{C \in S/\gamma} v(C), \text{ for all } S \subseteq N.$$

Myerson (1977) characterized this allocation rule in terms of *component efficiency* (for all $C \in N/\gamma$, $\sum_{i \in C} \mu_i(N, v, \gamma) = v(C)$) and *fairness* (for each $l = \{i, j\} \in \gamma$, $\psi_i(N, v, \gamma) - \psi_i(N, v, \gamma \setminus \{l\}) = \psi_j(N, v, \gamma) - \psi_j(N, v, \gamma \setminus \{l\})$). He also characterized it (Myerson, 1980) in terms of component efficiency and *balanced contributions* (given $i, j \in N$, $\psi_i(N, v, \gamma) - \psi_i(N, v, \gamma_{-j}) = \psi_j(N, v, \gamma) - \psi_j(N, v, \gamma_{-i})$).

3. Weighted Myerson values

In this section we admit several interpretations for weights in a graph and we accordingly define several weighted graph restricted games for players involved in a TU game with restrictions in the connections given by such a weighted network. Finally, we propose some values that generalize the classical Myerson one.

3.1. Weighted graphs

Definition 3.1. A weighted graph or a weighted network is a pair (N, γ_w), $N = \{1, \dots, n\}$ being a set of nodes and $\gamma_w = \{\gamma, \{w_l\}_{l \in \gamma}\}$,

[2] Let us observe that in the case $S = R = N$ a minimal connection subgraph of N in $(N, \gamma_{|N} = \gamma)$ is a spanning tree of (N, γ).