European Journal of Operational Research 238 (2014) 786–796

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Order monotonic solutions for generalized characteristic functions ☆

René van den Brink [a,*], Enrique González-Arangüena [b], Conrado Manuel [b], Mónica del Pozo [c]

[a] Department of Econometrics, Tinbergen Institute, VU University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands
[b] Dpto. de Estadística e I.O. III., Facultad de Estudios Estadísticos, Universidad Complutense de Madrid, Avda. Puerta de Hierro s/n, 28040 Madrid, Spain
[c] Dpto. de Economía, Universidad Carlos III de Madrid, Spain

ARTICLE INFO

Article history:
Received 5 August 2013
Accepted 8 April 2014
Available online 30 April 2014

Keywords:
Game theory
Cooperative TU-game
Generalized characteristic function
Order monotonicity

ABSTRACT

Generalized characteristic functions extend characteristic functions of 'classical' TU-games by assigning a real number to every ordered coalition being a permutation of any subset of the player set. Such generalized characteristic functions can be applied when the earnings or costs of cooperation among a set of players depend on the order in which the players enter a coalition.

In the literature, the two main solutions for generalized characteristic functions are the one of Nowak and Radzik (1994), shortly called NR-value, and the one introduced by Sánchez and Bergantiños (1997), shortly called SB-value. In this paper, we introduce the axiom of *order monotonicity* with respect to the order of the players in a unanimity coalition, requiring that players who enter earlier should get not more in the corresponding (ordered) unanimity game than players who enter later. We propose several classes of order monotonic solutions for generalized characteristic functions that contain the NR-value and SB-value as special (extreme) cases. We also provide axiomatizations of these classes.

1. Introduction

Generalized characteristic functions on a player set N, introduced by Nowak and Radzik (1994), extend characteristic functions of 'classical' TU-games by assigning a real number to every ordered coalition of N. Here an ordered coalition of N is a permutation of any subset of N. Classical characteristic functions form the special class where the real number or worth assigned to any ordered coalition only depends on the players that are part of this ordered coalition, i.e. it does not matter in which order the players enter the coalition. Generalized characteristic functions can be used in situations where the worth (or cost) that can be generated by a set of players depends on the order in which the players enter.

Consider, for example, the airport games of Littlechild and Owen (1973) to allocate the building and maintenance costs of airport landing strips, see also Littlechild and Thompson (1977). An airport cost situation consists of a set of airplanes (being the players in the game) and for each airplane a nonnegative cost of the airline strip that is necessary for this airplane to land. Since the airplanes are different they need landing strips of different length. In the associated airport game, the worth of a coalition (being a subset of the set of airplanes N) is the cost of the airline strip needed for the largest airplane in this coalition (assuming that larger airplanes need longer and more expensive landing strips).[1] But this means that building a landing strip of a certain size does not depend on the order in which the airplanes enter the coalition. The worth (cost) of a coalition is always fully determined by the cost for the largest airplane in the coalition. However, in real life construction industry it is usually more expensive to build a project in several steps than to build it fully at once. For example, when one wants to extend an existing landing strip then all the machinery has to be brought back to the airport, everything needs to be setup again, maybe some reconstruction or preparation needs to be done before being able to extend the existing landing strip. Then it would have been less costly to have built the longer landing strip at once. Therefore, instead of modelling an airport cost problem on n airplanes by an n-dimensional cost vector $c \in \mathbb{R}^N_+$ which ith component is the cost of building an airline strip suitable for airplane i, it seems more realistic to model it by an $n \times n$ dimensional cost matrix C, which first column coincides with the above mentioned cost vector c (i.e. the first column gives the cost for building the airline strip for i when there is nothing built yet), and which ijth component

☆ This research was supported by the "Plan Nacional de I+D+i" of the Spanish Government under the project MTM2008-06778-C02-02/MTM. We thank three anonymous referees for their valuable comments.

* Corresponding author. Tel.: +31 20 5985483; fax: +31 20 5986020.
E-mail addresses: jrbrink@feweb.vu.nl (R. van den Brink), egaran@estad.ucm.es (E. González-Arangüena), conrado@estad.ucm.es (C. Manuel), mopozoj@eco.uc3m.es (M. del Pozo).

[1] Instead of airplanes, in an airport game every airplane landing is a player but for convenience we simply call these airplanes.

http://dx.doi.org/10.1016/j.ejor.2014.04.016
0377-2217/

$c_{ij}, i \in \{1, \ldots, n\}, j \in \{2, \ldots, n\}$, is the cost of building (extending) an airline strip suitable for airplane i when the landing strip is built already for airplane $j-1$ (and smaller airplanes). So, we might consider the first column as a standard airport cost problem.

Example 1.1. Suppose that there are three airplanes $N = \{1, 2, 3\}$, where the costs of building an airline strip for airplane $i \in N$ is given by cost vector $c = (c_i)_{i \in N} = (1, 3, 4)$. However, it can be that extending the airline strip for airplane 1 to one for airplane 2 costs 3 additional to the cost made to build the already existing airline strip. If, further extending the airline strip for airplane 1 to one for airplane 3 costs 3, and extending the airline strip for airplane 2 to one for airplane 3 costs 2, this can be represented by cost matrix

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 4 & 3 & 2 \end{bmatrix}.$$

Such an airport cost problem, where the cost of the landing strip depends on the order in which airplanes announce that they want to use the landing strip, cannot be modelled by a classical TU-game. Although the entrance of an airplane that is not larger than the largest airplane in the coalition does not change the worth, when an airplane larger than the largest airplane in the coalition enters, then the additional cost depends on which is the largest airplane in the already existing coalition. Such *generalized airport cost problems* can be modeled by a generalized characteristic function where the worth of an ordered coalition of airplanes is the cost of the landing strip if it would be built sequentially, in each step being large enough to allow the corresponding airplane to land.

In the literature, the two main solutions for generalized characteristic functions are the one of Nowak and Radzik (1994), which we will call the NR-value, and the one introduced by Sánchez and Bergantiños (1997) and generalized in Sánchez and Bergantiños (2001), which we refer to as the SB-value. Both solutions extend the Shapley value (Shapley, 1953) of classical TU-games in the sense that for a generalized characteristic function that represents a classical characteristic function they yield the Shapley value of that classical game.[2]

In this paper we criticize some axioms underlying these two solutions and we propose a new axiom, called order monotonicity, and use it to characterize a class of solutions that contains the NR-value and the SB-value as extreme cases. Moreover, we revisit the parametric family of geometric solutions for generalized characteristic functions introduced in del Pozo, Manuel, González-Arangüena, and Owen (2011) and we characterize it.

The paper is organized as follows. Section 2 contains preliminaries on games in generalized characteristic function and games with a permission structure. In Section 3 we introduce the new axiom of order monotonicity and use it to define a new class of solutions for games in generalized characteristic functions. In Section 4 we discuss the special class of the so-called geometric solutions. Finally, Section 5 contains concluding remarks.

[2] The Shapley value is one of the most relevant solutions concept in cooperative game theory. Kamijo and Kongo (2012) compare this value with some other relevant ones. Moretti and Patrone (2008) is an excellent survey on the transversality of the Shapley value. Extensions of the Shapley value also have been considered in other generalizations of the standard TU-game model, for example for fuzzy games in e.g. Tsurumi, Tanino, and Inuiguchi (2001) and Li and Zhang (2009), games in partition function form in e.g. Grabisch and Funaki (2012), TU-games with awards in Lorenzo-Freire, Alonso-Meijide, Casas-Méndez, and Hendrickx (2007), and games where cooperation is restricted by certain combinatorial structures such as communication structures in Myerson (1980), probabilistic communication situations in Gómez, González-Arangüena, Manuel, and Owen (2008), convex geometries in Bilbao (1998), antimatroids in Algaba, Bilbao, van den Brink, and Jiménez-Losada (2003) and augmenting systems in Bilbao and Ordóñez (2009).

2. Preliminaries

2.1. Games and generalized games

A situation in which a finite set of players $N \subset \mathbb{N}$ can generate certain payoffs by cooperation can be described by a *cooperative game in characteristic function form* (also known as cooperative game with transferable utility or simply TU-game) being a pair $(N, \hat{v})$ where the characteristic function $\hat{v} : 2^N \to \mathbb{R}$ is a real function defined on 2^N (the set of all subsets of N), that satisfies $\hat{v}(\emptyset) = 0$. For each *coalition* $S \in 2^N$, the *worth* $\hat{v}(S)$ represents the (transferable) utility that players in S can obtain if they decide to cooperate. When there is no ambiguity with respect to the players set N, we will identify the game $(N, \hat{v})$ with its characteristic function $\hat{v}$. In the sequel we will denote the cardinality of coalitions $S, T, R \in 2^N$ by lower case s, t, r. We will denote by G^N the set of all characteristic functions with player set N. It is well-known that G^N is a $2^n - 1$ dimensional vector space, $n = |N|$, with the *unanimity games* $\{\hat{u}_S\}_{\emptyset \neq S \subseteq N}$ as basis. For every $S \subseteq N, S \neq \emptyset$, the unanimity game $\hat{u}_S$ is defined by $\hat{u}_S(T) = 1$ if $S \subseteq T$, and $\hat{u}_S(T) = 0$, otherwise. For a given $\hat{v} \in G^N$, the unanimity coefficients (i.e. the coordinates of $\hat{v}$ in the unanimity basis) $\{\Delta_{\hat{v}}(T)\}_{\emptyset \neq T \subseteq N}$ are given by (see Harsanyi (1959))

$$\Delta_{\hat{v}}(T) = \sum_{R \subseteq T} (-1)^{t-r} \hat{v}(R), \quad \emptyset \neq T \subseteq N. \tag{2.1}$$

A *solution* or *value* for TU-games is a function which assigns a payoff vector $x \in \mathbb{R}^N$ to every TU-game in G^N. One of the most famous solutions is the *Shapley value* (Shapley, 1953), φ^{Sh}, which is given by:

$$\varphi_i^{Sh}(N, \hat{v}) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} (\hat{v}(S \cup \{i\}) - \hat{v}(S)), \quad \text{for all } i \in N.$$

Nowak and Radzik (1994) introduced the concept of game in generalized characteristic function form where the order in which a coalition is formed influences the worth that can be generated. For each $S \in 2^N \setminus \{\emptyset\}$, let $\Pi(S)$ denote the set of all permutations or ordered coalitions of the players in S and, for notational convenience, $\Pi(\emptyset) = \{\emptyset\}$. We denote $\Omega(N) = \{T \in \Pi(S) \mid S \subseteq N\}$ as the set of all ordered coalitions with players in N. A game in generalized characteristic function form is a pair $(N, v), N$ being the player set and $v : \Omega(N) \to \mathbb{R}$ a real function (the generalized characteristic function), defined on $\Omega(N)$ and satisfying $v(\emptyset) = 0$.

For each $S \subseteq N$, and for every ordered coalition $T \in \Pi(S)$, $v(T)$ represents the economic possibilities of the players in S if the coalition is formed following the order given by T.

Example 2.1. Consider the generalized airport cost problem of Example 1.1. The corresponding generalized characteristic function is $v(1) = 1$, $v(2) = v(21) = 3$, $v(3) = v(31) = v(32) = v(312) = v(321) = 4$, $v(12) = 1 + 3 = 4$, $v(13) = v(132) = 1 + 3 = 4$, $v(23) = v(213) = v(231) = 3 + 2 = 5$ and $v(123) = 1 + 3 + 2 = 6$.

We denote by $\mathcal{G}^N$ the set of all generalized characteristic functions with player set N, and $\mathcal{G} = \{(N, v) | N \subset \mathbb{N},\ v \in \mathcal{G}^N\}$. As in the case of games in G^N, we will sometimes identify the game with its characteristic function.

Given an ordered coalition $T \in \Omega(N)$, there exists $S \subseteq N$ such that $T \in \Pi(S)$. We will denote by $H(T) = S$ the set of players in the ordered coalition T, and $t = |H(T)|$ (if there is no confusion with $|T|$). Each ordered coalition $T = (i_1, \ldots, i_t) \in \Omega(N)$ establishes a strict linear order $\prec_T$ in $H(T)$, defined as follows: for all $i, j \in H(T), i \prec_T j$ (i precedes j in T) if and only if there exist $k, l \in \{1, \ldots, t\}, k < l$, such that $i = i_k, j = i_l$.