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Decision Support

Consumption–investment strategies with non-exponential discounting and logarithmic utility

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ABSTRACT

In this paper, we revisit the consumption–investment problem with a general discount function and a logarithmic utility function in a non-Markovian framework. The coefficients in our model, including the interest rate, appreciation rate and volatility of the stock, are assumed to be adapted stochastic processes. Following Yong (2012a,b)'s method, we study an N-person differential game. We adopt a martingale method to solve an optimization problem of each player and characterize their optimal strategies and value functions in terms of the unique solutions of BSDEs. Then by taking limit, we show that a time-consistent equilibrium consumption–investment strategy of the original problem consists of a deterministic function and the ratio of the market price of risk to the volatility, and the corresponding equilibrium value function can be characterized by the unique solution of a family of BSDEs parameterized by a time variable.

1. Introduction

In recent years, research on the time-inconsistent preferences has attracted an increasing attention. The empirical studies of human behavior reveal that the constant discount rate assumption is unrealistic, (see, for example, Thaler (1981), Ainslie (1992) and Loewenstein & Prelec (1992)). Experimental evidence shows that economic agents are impatient about choices in the short term but are more patient when choosing between long-term alternatives. Particularly, cash flows in the near future tend to be discounted at a significantly higher rate than those occur in the long run. Considering such behavioral feature, economic decisions may be analyzed using the hyperbolic discounting (see Phelps & Pollak (1968)). Indeed, the hyperbolic discounting has been widely adopted in microeconomics, macroeconomics, and behavioral finance, such as Laibson (1997) and Barro (1999) among others.

However, difficulties arise when one attempts to solve an optimal control problem with a non-constant discount rate by some standardized control techniques, such as dynamic programming approach. In fact, these techniques lead to time inconsistent strategies, i.e., a strategy that is optimal for the initial time may not be optimal later (see, for example, Ekeland & Pirvu (2008) and Yong (2012b)). In other words, the classical dynamic programming principle fails to solve the so-called time-inconsistent control problem. So, how to obtain a time-consistent strategy for time-inconsistent control problems becomes an interesting and challenging problem. In Strotz (1955), the author studies a cake eating problem within a game theoretic framework where the players are the agent and his/her future selves, and seek a subgame perfect Nash equilibrium point for this game. Strotz's work has been pursued by many others, such as Pollak (1968), Peleg and Yaari (1973), Goldman (1980) and Laibson (1997) among others.

Recently, the time inconsistent control problems regain considerable attention in the continuous-time setting. A modified HJB equation is derived in Marín-Solano and Navas (2010) which solves the optimal consumption and investment problem with non-constant discount rate for both naive and sophisticated agents. The similar problem is also considered by another approach in Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008), which provide the precise definition of the equilibrium concept in continuous time for the first time. They characterize the equilibrium policies through the solutions of a flow of BSDEs, and they show, with special form of the discount factor, this BSDE reduces to a system of two ODEs which has a solution. There are some literature following their definition of equilibrium strategy. In Björk and Murgoci (2010), the time-inconsistent control problem is considered in a general Markov framework, and an extended HJB equation together with the verification theorem are derived.

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Björk, Murgoci, and Zhou (2014) investigates the Markowitz's problem with state-dependent risk aversion by utilizing the extended HJB equation obtained in Björk and Murgoci (2010). Considering the hyperbolic discounting, Ekeland, Mbodji, and Pirvu (2012) studies the portfolio management problem for an investor who is allowed to consume and take out life insurance, and they characterize the equilibrium strategy by an integral equation.

Another approach to the time-inconsistent control problem is developed by Yong (2011, 2012a, 2012b). In Yong's papers, a sequence of multi-person hierarchical differential games is studied first and then the time-consistent equilibrium strategy and equilibrium value function are obtained by taking limit. A brief description of the method is given as follows. Let $T>0$ be the fixed time horizon and $t\in[0,T)$ be the initial time. Take a partition $\Pi=\{t_k|0\leqslant k\leqslant N\}$ of the time interval $[t,T]$ with $t=t_0<t_1<\cdots<t_N=T$, and with the mesh size

$$\|\Pi\|=\max_{1\leqslant k\leqslant N}(t_k-t_{k-1}).$$

Consider an N-person differential game: for $k=1,2,\ldots,N$, the k-th player controls the system on $[t_{k-1},t_k)$, starting from the initial state $(t_{k-1},X(t_{k-1}))$ which is the terminal state of the $(k-1)$-th player, and tries to maximize his/her own performance functional. Each player knows that the later players will do their best, and will modify their control systems as well as their cost functionals. In the performance functional, each player discounts the utility in his/her own way. Then for any given partition Π, a Nash equilibrium strategy is constructed to the corresponding N-person differential game. Finally, it can be shown that as the mesh size $\|\Pi\|$ approaches to zero, the Nash equilibrium strategy to the N-person differential game approaches to the desired time-consistent solution of the original time-inconsistent problem. By this method, Yong (2011, 2012a) considers a deterministic time-inconsistent linear-quadratic control problem. Considering a controlled stochastic differential equation with deterministic coefficients, Yong (2012b) investigates a time-inconsistent problem with a general cost functional and derives an equilibrium HJB equation.

In this paper, we revisit the consumption–investment problem (Merton, 1969, 1971) with a general discount function and a logarithmic utility function. In contrast to the references cited above, we consider this problem in a non-Markovian framework. More specifically, the coefficients in our model, including the interest rate, appreciation rate and volatility of the stock, are assumed to be adapted stochastic processes. To our best knowledge, the literature on the time-inconsistent problem in a non-Markovian model is rather limited. A time-inconsistent stochastic linear-quadratic control problem is studied in a model with random coefficients by Hu, Jin, and Zhou (2012). A time-consistent strategy is obtained for the mean–variance portfolio selection by Czichowsky (2013) in a general semimartingale setting. Following Yong's method, we first study an N-person differential game. Similar to Hu, Imkeller, and Müller (2005) and Cheridito and Hu (2011), we adopt a martingale method to solve an optimization problem of each player and characterize their optimal strategies and value functions in terms of the unique solutions of BSDEs. Then by taking limit, we show that a time-consistent equilibrium consumption–investment strategy of the original problem consists of a deterministic function and the ratio of the market price of risk to the volatility, and the corresponding equilibrium value function can be characterized by the unique solution of a family of BSDEs parameterized by a time variable which can be understood as the initial time of each player in the N-person differential game.

The remainder of this paper is organized as follows. Section 2 introduces the model. In Section 3, we study the N-person differential game. Section 4 gives a time-consistent equilibrium strategy and time-consistent equilibrium value function to the original problem. Section 5 concludes the paper. Some proofs and technical results are collected in the appendices.

2. The model

Let $T>0$ be a fixed finite time horizon, and $\{W(t)\}_{0\leqslant t\leqslant T}$ be a standard Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leqslant t\leqslant T},\mathsf{P})$. Here the filtration $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ is the augmentation under P of $\mathcal{F}_t^W:=\sigma(W(s),0\leqslant s\leqslant t),t\in[0,T]$. We consider a market which consists of a bond and a stock. The price of the bond evolves according to the differential equation

$$\begin{cases}\mathrm{d}B(s)=r(s)B(s)\mathrm{d}s, & s\in[0,T],\\ B(0)=1.\end{cases}$$

The price of the stock is modeled by the stochastic differential equation

$$\begin{cases}\mathrm{d}S(s)=\mu(s)S(s)\mathrm{d}s+\sigma(s)S(s)\mathrm{d}W(s), & s\in[0,T],\\ S(0)=s_0,\end{cases}$$

where $s_0>0$. The interest rate process $\{r(t)\}_{0\leqslant t\leqslant T}$ as well as the appreciation rate $\{\mu(t)\}_{0\leqslant t\leqslant T}$ and volatility $\{\sigma(t)\}_{0\leqslant t\leqslant T}$ of the stock are assumed to be $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$-adapted and bounded uniformly in $(t,\omega)\in[0,T]\times\Omega$. In addition, we require that the volatility process $\{\sigma(t)\}_{0\leqslant t\leqslant T}$ is bounded away from zero.

A consumption–investment policy is a bivariate process $(c(t),u(t))\in\mathbb{R}^+\times\mathbb{R}$, where $c(t)$ is the consumption rate at time t as a proportion of the wealth and $u(t)$ is the proportion of wealth invested in the stock at time t.

Let

$$\mathcal{C}[t,T]=\Big\{c:[t,T]\times\Omega\to\mathbb{R}^+|c(\cdot)\text{ is a predictable process for which }\int_t^T|c(s)|\mathrm{d}s<\infty,\ a.s.\Big\},$$
$$\mathcal{U}[t,T]=\Big\{u:[t,T]\times\Omega\to\mathbb{R}|u(\cdot)\text{ is a predictable process for which }\int_t^T|u(s)\sigma(s)|^2\mathrm{d}s<\infty,\ a.s.\Big\}.$$

For any initial time $t\in[0,T]$ and initial wealth $x>0$, applying a consumption–investment policy $(c(s),u(s))\in\mathcal{C}[t,T]\times\mathcal{U}[t,T]$, the wealth process of the investor, denoted by $X(\cdot)$, is governed by

$$\begin{cases}\mathrm{d}X(s)=[r(s)-c(s)+u(s)\sigma(s)\theta(s)]X(s)\mathrm{d}s+u(s)\sigma(s)X(s)\mathrm{d}W(s), & s\in[t,T],\\ X(t)=x,\end{cases}\tag{1}$$

where

$$\theta(s):=\frac{\mu(s)-r(s)}{\sigma(s)}.$$

To emphasize the dependence of the wealth process on the initial state and the policy, we also write the solution of (1) as $X(\cdot;t,x,c(\cdot),u(\cdot))$.

In this paper, we focus on the logarithmic utility function. At any initial time $t\in[0,T]$ with initial wealth $x>0$, the performance functional, i.e. the expected discounted utility from the consumption and terminal wealth is given by

$$J(t,x;c(\cdot),u(\cdot))=\mathsf{E}_t\left[\int_t^T h(s-t)\ln(c(s)X(s))\mathrm{d}s+h(T-t)\ln X(T)\right],$$

where $\mathsf{E}_t[\cdot]=\mathsf{E}[\cdot|\mathcal{F}_t]$ and $h(\cdot)$ is a general discount function satisfying

$$h(0)=1,\quad h(\cdot)>0,\quad h'(\cdot)\le 0,\quad \int_0^T h(s)\mathrm{d}s<\infty.$$

We also impose a technical assumption on h.