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Short Communication

Hedging Conditional Value at Risk with options

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ABSTRACT

We present a method of hedging Conditional Value at Risk of a position in stock using put options. The result leads to a linear programming problem that can be solved to optimise risk hedging.

1. Introduction

One of the natural ideas to reduce risk of a position in stock is to buy put options. By doing so one can cut off the undesirable scenarios, while leaving oneself open to the positive outcomes. A choice of a high strike price of the put option does cut off more of the unfavourable states, but at the same time produces higher hedging costs. The question of how to balance the two trends so that the level of risk measured by Value at Risk (VaR) is minimised was investigated by Ahn, Boudoukh, Richardson, and Whitelaw (1999).

The Value at Risk, which is the worst case scenario of loss an investment might incur at a given confidence level, has established its position as one of the standard measures of risk, and is widely used throughout the field of finance and risk management. One of its shortcomings is that it neglects potential severity of unlikely events. Another, that it is not sub-additive, and is thus not a coherent risk measure (Artzner, Delbaen, Eber, & Heath, 1999). Its most common modification to achieve these goals is the Conditional Value at Risk (CVaR) (also referred to as 'Expected Shortfall'), which takes into the account the average loss exceeding VaR. The CVaR is a coherent risk measure (the proof can be found in the work of Acerbi & Tasche, 2002).

In this paper we show a mirror result to Ahn et al. (1999), using CVaR instead of VaR. It turns out that in such setting one can achieve closed form formulae for CVaR of stock hedged with puts. These can be used to optimise the position by solving a linear programming problem.

We restrict our attention to the Black–Scholes model and consider investments in stock and put options. The optimisation of CVaR can be carried out under more general assumptions, using also other securities (as an example see Rockafellar & Uryasev, 2000, 2002). One can also hedge CVaR dynamically (as in the work of Melnikov & Smirnov, 2012), which provides slightly better results. Dynamic strategies though require constant rebalancing, which in practice can be costly. Advantages of our approach are as follows: its simplicity; closed form analytic formula for CVaR; protection against risk is very similar to the one attainable using dynamic strategies.

The paper is organised as follows. Section 2 recalls the results of Ahn et al. (1999) for hedging of VaR with put options. This section serves also as preliminaries to the paper. In Section 3 we generalise the result to use CVaR instead of VaR. The main result of the paper is given in Theorem 4. The section ends with an example of its application. In Section 4 we compare our method to the results attainable using dynamic strategies. They turn out to be close. We finish the paper with a short conclusion in Section 5.

2. Hedging Value at Risk

In this section we set up our notations and recall the results of Ahn et al. (1999).

Let X be a random variable, which represents a gain from an investment. For α in $(0, 1)$, we define the *Value at Risk* of X, at confidence level $1-\alpha$, as $\mathrm{VaR}^{\alpha}(X) = -q^{\alpha}(X)$, where $q^{\alpha}(X)$ is the upper α-quantile of X.

We consider the Black–Scholes model, where the stock price evolves according to $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$, with the money market account $dA(t) = rA(t)dt$. A European put option with strike price K and maturity T has payoff $P(T) = (K - S(T))^{+}$ and costs

$$P(0) = P(r, T, K, S(0), \sigma) = Ke^{-rT}N(-d_-) - S(0)N(-d_+), \tag{1}$$

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where

$$d_+ = d_+(r, T, K, S(0), \sigma) = \frac{\ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad (2)$$

$$d_- = d_-(r, T, K, S(0), \sigma) = d_+ - \sigma\sqrt{T},$$

and N is the standard normal cumulative distribution function.

Assume that we buy x shares of stock and z_i put options with strikes K_i, which cost $P_i(t)$ for $i = 1, \ldots, n$ and $t = 0, T$. Let $\mathbf{z}$, $\mathbf{1}$ and $\mathbf{P}(t)$ be vectors in $\mathbb{R}^n$ defined as

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \qquad \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad \mathbf{P}(t) = \begin{bmatrix} P_1(t) \\ \vdots \\ P_n(t) \end{bmatrix}.$$

The value of our investment at time t is $V_{(x,\mathbf{z})}(t) = xS(t) + \mathbf{z}^T\mathbf{P}(t)$. The following theorem can be used to compute VaR for the discounted gain

$$X_{(x,\mathbf{z})} = e^{-rT}V_{(x,\mathbf{z})}(T) - V_{(x,\mathbf{z})}(0).$$

Theorem 1 (Ahn et al., 1999). *If $z_i \geq 0$, for $i = 1, \ldots, n$, and $\mathbf{z}^T\mathbf{1} \leq x$, then*

$$\mathrm{VaR}^\alpha(X_{(x,\mathbf{z})}) = V_{(x,\mathbf{z})}(0) - e^{-rT}(xq^\alpha(S(T)) - \mathbf{z}^T\mathbf{q}^\alpha(-\mathbf{P}(T))), \quad (3)$$

where

$$\mathbf{q}^\alpha(-\mathbf{P}(T)) = -\begin{bmatrix} (K_1 - q^\alpha(S(T)))^+ \\ \vdots \\ (K_n - q^\alpha(S(T)))^+ \end{bmatrix}. \quad (4)$$

3. Hedging Conditional Value at Risk

One of the shortcomings of VaR is that it neglects the tail of the loss distribution. An improvement in this respect is the *Conditional Value at Risk*, defined as

$$\mathrm{CVaR}^\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}^\beta(X)d\beta = -\frac{1}{\alpha}\int_0^\alpha q^\beta(X)d\beta,$$

with a well known equivalent form

$$\mathrm{CVaR}^\alpha(X) = -\frac{1}{\alpha}[\mathbb{E}(X\mathbf{1}_{\{X\leq q^\alpha(X)\}}) + q^\alpha(X)(\alpha - \mathbb{P}(X \leq q^\alpha(X))]. \quad (5)$$

The CVaR also has the advantage of being a coherent risk measure (Acerbi & Tasche, 2002; Artzner et al., 1999).

Our aim is to give a mirror result to Theorem 1, using CVaR as the risk measure. We start with a simple lemma.

Lemma 2. *For any $q \in \mathbb{R}$,*

$$\mathbb{E}\left(S(T)|W(T) \leq q\sqrt{T}\right) = \frac{1}{N(q)}S(0)e^{\mu T}N\left(q - \sigma\sqrt{T}\right).$$

Proof. Let $Z = W(T)/\sqrt{T}$. Since $\mathbb{P}(Z \leq q) = N(q) > 0$,

$$\begin{aligned}\mathbb{E}(S(T)|Z \leq q) &= \frac{1}{P(Z \leq q)}\int_{-\infty}^q S(0)e^{((\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x)}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\ &= \frac{1}{N(q)}S(0)e^{\mu T}\int_{-\infty}^q \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}dx \\ &= \frac{1}{N(q)}S(0)e^{\mu T}N\left(q - \sigma\sqrt{T}\right),\end{aligned}$$

as required. □

Let Z be a random variable with standard normal distribution $N(0, 1)$. To compute $\mathrm{CVaR}^\alpha(X_{(x,\mathbf{z})})$, we introduce notations

$$d_-^\mu = d_-(\mu, T, K, S(0), \sigma), \quad d_+^\mu = d_-^\mu + \sigma\sqrt{T},$$
$$d_-^{\mu,\alpha} = \max(d_-^\mu, -q^\alpha(Z)), \quad d_+^{\mu,\alpha} = d_-^{\mu,\alpha} + \sigma\sqrt{T},$$

$$P^\alpha(K) = Ke^{-\mu T}N(-d_-^{\mu,\alpha}) - S(0)N(-d_+^{\mu,\alpha}). \quad (6)$$

We first consider the case when we invest in puts with a single strike $K_1 = K$.

Proposition 3. *If $\mathbf{z} = [z_1]$, for $z_1 = z \in [0, x]$, then*

$$\mathrm{CVaR}^\alpha(X_{(x,\mathbf{z})}) = V_{(x,\mathbf{z})}(0) - \frac{1}{\alpha}e^{(\mu-r)T}[xS(0)N(q^\alpha(Z) - \sigma\sqrt{T}) + zP^\alpha(K)].$$

Proof. We first observe that

$$X_{(x,\mathbf{z})} = e^{-rT}(xS(T) + z(K - S(T))^+) - V_{(x,\mathbf{z})}(0). \quad (7)$$

Since $z \leq x$, we see that

$$s \to e^{-rT}(xs + z(K - s)^+) - V_{(x,\mathbf{z})}(0) \quad (8)$$

is a non-decreasing function of s. Also $\xi \to S(0)\exp((\mu - \sigma^2/2)T + \sigma\sqrt{T}\xi)$ is increasing. Combining these two facts, taking $Z = W(T)/\sqrt{T}$,

$$\{X_{(x,\mathbf{z})} \leq q^\alpha(X_{(x,\mathbf{z})})\} = \{S(T) \leq q^\alpha(S(T))\} = \{Z \leq q^\alpha(Z)\}. \quad (9)$$

We first prove the claim for $z < x$. Then (8) is strictly increasing, therefore $\mathbb{P}(X_{(x,\mathbf{z})} \leq q^\alpha(X_{(x,\mathbf{z})})) = \mathbb{P}(S(T) \leq q^\alpha(S(T))) = \alpha$, and

$$\begin{aligned}\mathrm{CVaR}^\alpha(X_{(x,\mathbf{z})}) &= -\mathbb{E}(X_{(x,\mathbf{z})}|X_{(x,\mathbf{z})} \leq q^\alpha(X_{(x,\mathbf{z})})) \\ &= -\mathbb{E}(X_{(x,\mathbf{z})}|Z \leq q^\alpha(Z)) \qquad (\text{see } (9)) \\ &= V_{(x,z)}(0) - e^{-rT}x\mathbb{E}(S(T)|Z \leq q^\alpha(Z)) \quad (\text{see } (7)) \\ &\quad - e^{-rT}z\mathbb{E}((K - S(T))^+|Z \leq q^\alpha(Z)). \end{aligned} \quad (10)$$

We now compute the last term in (10). Since $\{S(T) \leq K\} = \{Z \leq -d_-^\mu\}$,

$$\begin{aligned}&\mathbb{E}((K - S(T))^+|Z \leq q^\alpha(Z)) \\ &= \frac{1}{\alpha}\int_{-\infty}^{\min(q^\alpha(Z),-d_-^\mu)}\left(K - S(0)e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x}\right)\frac{1}{\sqrt{2\pi}}e^{-x^2}dx \\ &= \frac{1}{\alpha}\int_{-\infty}^{-d_-^{\mu,\alpha}} K\frac{1}{\sqrt{2\pi}}e^{-x^2}dx - \frac{1}{\alpha}\int_{-\infty}^{-d_-^{\mu,\alpha}} S(0)e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x} \\ &\quad\times \frac{1}{\sqrt{2\pi}}e^{-x^2}dx \\ &= \frac{1}{\alpha}KN(-d_-^{\mu,\alpha}) - \frac{1}{\alpha}\mathbb{P}(Z \leq -d_-^{\mu,\alpha})\mathbb{E}(S(T)|Z \leq -d_-^{\mu,\alpha}) \\ &= \frac{1}{\alpha}KN(-d_-^{\mu,\alpha}) - \frac{1}{\alpha}S(0)e^{\mu T}N(-d_-^{\mu,\alpha} - \sigma\sqrt{T}) \quad (\text{by Lemma 2}) \\ &= \frac{1}{\alpha}e^{\mu T}(Ke^{-\mu T}N(-d_-^{\mu,\alpha}) - S(0)N(-d_+^{\mu,\alpha})).\end{aligned}$$

Substituting the above into (10) and applying Lemma 2 gives the claim.

We now need to consider the case when $z = x$. Since for any $\beta \in (0, 1)$, $\lim_{z\nearrow x} q^\beta(X_{(x,\mathbf{z})}) = q^\beta(X_{(x,x)})$, we obtain

$$\begin{aligned}\lim_{z\nearrow x}\mathrm{CVaR}^\alpha(X_{(x,\mathbf{z})}) &= \lim_{z\nearrow x}\frac{-1}{\alpha}\int_0^\alpha q^\beta(X_{(x,\mathbf{z})})d\beta \\ &= \frac{-1}{\alpha}\int_0^\alpha q^\beta(X_{(x,x)})d\beta = \mathrm{CVaR}^\alpha(X_{(x,x)}).\end{aligned}$$

Hence the result follows from the fact that the formula for $\mathrm{CVaR}^\alpha(X_{(x,\mathbf{z})})$ in the claim is continuous with respect to z. □

We can now formulate our main result.