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Decision Support

Cost allocation in asymmetric trees

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ABSTRACT

Agents are connected each other through a tree. Each link of the tree has an associated cost and the total cost of the tree must be divided among the agents. In this paper we assume that agents are asymmetric (think on countries that use aqueducts to bring water from the rainy regions to the dry regions, for example). We suppose that each agent is entitled with a production and demand of a good that can be sent through the tree. This heterogeneity implies that the links are not equally important for all the agents. In this work we propose, and characterize axiomatically, two rules for sharing the cost of the tree when asymmetries apply.

1. Introduction

The study on the structure of service networks (such as power grids, gas pipelines, telecommunications or transportation infrastructures) has become more and more important. Those networks are costly (some times that cost is the amount to pay for its construction, and some other times it refers to the needed money for its maintenance). Quite often the structure of the network is a tree, because it allows to have all agents connected in the cheapest way. The problem of allocation the cost of a network, and particularly of a tree, has been widely studied in the literature.

Let us consider the simplest problem we may define, that is, two agents with a link connecting them. To divide the cost of the link equally between the two extremes, is a quite reasonable outcome when there is no more information and agents are assumed to be homogeneous. However, there are many cases where that homogeneity does not apply, and the link is much more important for one of the agents than for the other. The electrical interconnection between France and Spain is a quite illustrative example of this situation. These two countries decided to communicate their national power grid with a link crossing the Pyrenees. The cost of this connection has to be split between both countries, taking into account that Spain will obtain more beneficial than France from the construction of the link. This is because France produces enough electricity to cover, in case of failure in Spain, both its own demand and the Spanish one. The question then arises: How should the cost of the new connection be distributed between France and Spain considering all the elements? Similar questions emerge in the construction of the gas pipelines between Europe and Africa or Russia and Germany. Another situation that reflects this asymmetry is the network of aqueducts and pipelines that bring water from the rainy regions to the arid areas. The first ones use to have enough water to cover its demand and to even send the surplus to where it is needed, whereas the arid regions hardly have enough resource to fulfilled its own demand. Until which point is fair to oblige to the rainy areas to contribute to the cost of the pipelines? This paper deals with these situations, where the agents in the tree are asymmetric. This asymmetry will be induced by different demand and production for different agents.

In its general formulation, we consider a set of agents, each of them able to produce and demand a service like water. All agents are connected to each other forming a tree. Through this tree agents can send and receive water (or any other good). Hence, if the production of one agent fails, its demand can still be satisfied by obtaining the water from another agent. This can be obviously done only if the production of the other agent is high enough to compensate the failure. We assume that the tree is given and that each link of the tree has an associated cost that should be paid by the agents. This cost could be interpreted as a construction cost, maintenance cost (the tree already exists there but there are recurring maintenance costs, otherwise cannot be operative) or usage cost (the tree is there but we must pay a cost for using it). To sum up, a *problem* has five elements: a set of agents, a tree that describes the communication structure, a cost function that sets the cost of each link, a vector of demands (one demand for each agent), and a vector of productions (one production for each agent). A *rule* is a mechanism to distribute the cost of the tree among the agents.

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Our paper belongs to literature on the axiomatic analysis of cost allocation rules. Some papers of this literature focus on general problems like Moulin and Shenker (1992), Sprumont (1998), and Tijs and Driessen (1986). Other papers study some particular situations associated with networks, like Estevez-Fernandez (2012), Moulin and Laigret (2011), Norde, Fragnelli, Garcia-Jurado, Patrone, and Tijs (2002). Other papers focus on situations associated with a tree, like Bird (1976), Bogomolnaia, Holzman, and Moulin (2010), Littlechild and Owen (1973), and Ni and Wang (2007).

The particular case of trees has been a focal point (mostly because it is the most efficient way to communicate all the agents, regardless other aspects like flows and capacities). In the so-called minimum cost spanning problem several agents (nodes) need a good (water, for example) which can only be provided by a special node called the source (a water tank, for example). Thus, agents have to be connected to the source. Bird (1976), probably the seminal paper on this field, studies how to share the cost of a given tree among its nodes. Several papers came after in this field, proposing new solutions and analyzing the properties such solutions may satisfy. We can mention, for instance, the papers of Kar (2002), Dutta and Kar (2004), Tijs, Branzei, Moretti, and Norde (2006), Bergantiños and Vidal-Puga (2007, 2010), Bergantiños and Lorenzo-Freire (2008), Bogomolnaia and Moulin (2010), Bergantiños, Lorenzo, and Lorenzo-Freire (2011), Hougaard and Tvede (2012) and Trudeau (2012). In a parallel way other papers propose rules in problems arising from extensions of the minimum cost spanning tree problem. For instance, Fernandez, Hinojosa, and Puerto (2004) study multicriteria cost spanning tree games; Bergantiños and Gómez-Rúa (2010) study situations where agents are grouped; Dutta and Mishra (2012) study situations where costs are asymmetric; and Bergantiños, Gómez-Rúa, Llorca, Pulido, and Sánchez-Soriano (2012, 2014) study situations where the computation of the optimal tree is *NP* hard. In this paper we consider situations where each node can provide the service. Thus each node has a demand and a capacity production. As far as we are aware no other work has studied the problem presented in this paper. A real example captured by our model, but not by the previous ones, is the transfer of water from Tagus river to Segura river in Spain.

Our paper belongs to literature on the axiomatic analysis of cost allocation rules in networks. See Sharkey (1995) and Thomson (2001) for two surveys of this literature. In the axiomatic method, rules are justified in terms of the axioms they fulfill. In general, suitable combinations of desirable axioms are used to differentiate among rules. Thus, we introduce a collection of axioms that are adequate for the framework we study.

The first group of axioms are: *Fairness for two agents* says that, when there are only two agents and one link between them, the cost of the connection is paid by the agent that gets profit from its existence. *Symmetry* states that symmetric agents should pay the same. *Independence reallocation* requires that if the cost of connecting two individuals is zero, both cannot benefit from reallocating productions and demands between themselves. *Network-cost independence of extra costs* refers to how to allocate punctual increments of the cost of a single link. Our first result identifies the rule that satisfies these four axioms. We call it the *equal division across components rule*, which works as follows. For each link l, if we remove l from the tree we have two connected components. The cost of l is divided among the components following the idea of fairness for two agents, but applied to each component. We compute the aggregate production and demand of both components. If the link is profitable for only one component, then this component pays the cost. If the link is profitable for both or nobody, then each component pays half of the cost. Once this is done, the payment of each component is equally allocated among the agents belonging to it. We apply the same reasoning to all the links of the tree and the contribution of an agent is the sum of his contributions in the removal of all links.

The second group of axioms we propose is the following. *Cost additivity* simply says that the rule is additive with respect to the cost function. *Stand alone core* states that the rule must select allocation within the core (whenever the core is not empty). An agent is *safe* in a tree when, in case of failure of his production, the other agents can fulfill its demand. After the removal of a link in the tree, the safe status of an agent may change. *Balanced contributions with respect to the safe status* requires that if the cost of a link increases, all agents with the same status with respect to such a link should be affected in the same way. In our second result we characterize the rule that fulfills the three previous properties. We call it *the equal safety rule*. This rule also specifies how to divide the cost of each single link. Now, the cost of a link is equally split among all the agents for whom such a link is necessary for their safety. If some link is not necessary for any agent then, the cost of the link is equally divided among all the agents. Again, the contribution of an agent is the sum of his contributions to all links of the tree.

We apply both rules to the design of a tariff system for the water transferred from Tagus river to Segura river in Spain. Even both rules are different, in this particular case coincide.

The rest of the paper is structured as follows. In Section 2 we present the model and the elements of the problem. In Section 3 we introduce the axioms we use in the rest of the paper. Sections 4 and 5 are devoted to the characterizations we aforementioned. Finally, Section 6 concludes.

2. Model

Let $U = \{1, 2, 3, \ldots\}$ be the (infinite) set of possible **agents** and $N \subset U$. Usually we take $N = \{1, 2, \ldots, n\}$ where $n = |N|$. A **network** g is a collection of unordered pairs in N, i.e., $g = \{\{i,j\} : \{i,j\} \subset N\}$. When there is no room for confusion we denote the elements of g simply as ij instead of $\{i,j\}$. The agents $i \in N$ involved in the network are called **nodes**, while the pairs ij are called **links**. Given a network g, the set of links and nodes of g are denoted by $L(g)$ and $N(g)$ respectively. Given $S \subset N, g_S$ denotes the restriction of g to S, namely $g_S = \{ij \in g : ij \subset S\}$. Given a network g and a link $ij \in g, g \setminus ij$ denotes the network resulting from dropping the link ij from g.

A **path in** g **between** i **and** j is a sequence of links in g that starts in node i and finishes in node j, i.e., it is a string $k_1k_2, \ldots, k_{h-1}k_h$ such that $k_q \neq k_r$ for all $q, r \in \{1, \ldots, h\}, k_{q-1}k_q \in g$ for all $q \in \{2, \ldots, h\}, i = k_1$, and $j = k_h$. We say that i **and** j **are connected in** g if there exists a path in g between i and j. We say that $S \subset N$ is a **connected component** in g if any pair of agents in S are connected whereas no agent in S is connected with an agent in $N \setminus S$. Let N/g denote the partition of N in connected components. Given $i \in N$, we denote by A_i^g the connected component of N/g to which i belongs to.

A **cycle in** g is a path in g between i and i different from ii. A **forest** is a network without cycles. A **tree** is a forest in which each pair of nodes are connected.

If g is a forest then N/g could have several elements. If g is a tree then $N/g = \{N\}$. Given a tree t and $ij \in t, N/(t \setminus ij)$ has two connected components. We denote by A_i^{ij} $\left(\text{respectively } A_j^{ij}\right)$ the component to which i (respectively j) belongs to.

A **cost function** is a mapping $c : N \times N \to \mathbb{R}_+$ where $c(ij)$ is the cost associated to link ij.

We assume that the cost of connecting agent i with itself is null, $c(ii) = 0$ for each $i \in N$; and connecting i with j is as costly as connecting j with i, this is, $c(ij) = c(ji)$ for each $\{i,j\} \subseteq N$. For each network t, abusing the notation we denote by $c(t)$ the cost of all links in $t, c(t) = \sum_{l \in t} c(l)$.