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Invited Review

On some applications of the selective graph coloring problem

Marc Demange [a,d], Tınaz Ekim [b,*], Bernard Ries [c,d], Cerasela Tanasescu [a,e]

[a] ESSEC Business School, Paris, France
[b] Bogazici University, Istanbul, Turkey
[c] PSL, Université Paris-Dauphine, 75775 Paris Cedex 16, France
[d] CNRS, LAMSADE UMR 7243, France
[e] CEREGMIA, University of Antilles-Guyane, Schoelcher, Martinique, France

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ABSTRACT

In this paper we present the *Selective Graph Coloring Problem*, a generalization of the standard graph coloring problem as well as several of its possible applications. Given a graph with a partition of its vertex set into several *clusters*, we want to select one vertex per cluster such that the chromatic number of the subgraph induced by the selected vertices is minimum. This problem appeared in the literature under different names for specific models and its complexity has recently been studied for different classes of graphs. Here, we describe different models – some already discussed in previous papers and some new ones – in very different contexts under a unified framework based on this graph problem. We point out similarities between these models, offering a new approach to solve them, and show some generic situations where the selective graph coloring problem may be used. We focus on specific graph classes motivated by each model, and we briefly discuss the complexity of the selective graph coloring problem in each one of these graph classes and point out interesting future research directions.

1. Introduction

1.1. Definitions and motivation

Graph coloring is one of the most studied optimization problems in graph theory. Given a simple graph, it consists in assigning one color to each vertex such that two vertices linked by an edge get different colors and the total number of colors used is minimized. It is widely used to model several types of real applications such as scheduling, timetabling, frequency allocation, wavelength routing and many more (see for instance Dandashi & Al-Mouhamed, 2010; Gamache, Hertz, & Ouellet, 2007; Giaro, Kubale, & Obszarski, 2009; Qu, Burke, & McCollum, 2009; Talaván & Yáñez, 2008). Several generalizations of the classical graph coloring problem are also considered in the literature to cover an even wider range of applications with various constraints (see for instance Demange, Ekim, & de Werra, 2009; Sotskov, Dolgui, & Werner, 2001). In this paper, we motivate a further generalization of the usual graph coloring problem which introduces more flexibility to the applications by offering the possibility of choosing among several predefined strategies.

In this paper, all graphs $G = (V, E)$ are undirected and simple. A *stable set* is a set of vertices that are pairwise non-adjacent and a *clique* is a set of vertices that are pairwise adjacent. We denote by $\alpha(G)$ and $\omega(G)$ the size of a maximum stable set and the size of a maximum clique in G, respectively. The complementary graph $\bar{G} = (V, \bar{E})$ of $G = (V, E)$ is defined by $uv \in \bar{E} \iff u \neq v$ and $uv \notin E$. An induced path on k vertices is denoted by P_k. The graph obtained by taking k disjoint copies of G (with no edges between any two copies) is referred to as kG. For $V' \subseteq V$, $G[V']$ denotes the subgraph induced by V' in G.

A *k-coloring* of G is a mapping $c: V \to \{1, \ldots, k\}$ ($c(u)$ is called the *color* of vertex u) such that $c(u) \neq c(v)$ for all $uv \in E$ (each color class defined by the set of vertices with a same color forms a stable set). The smallest integer k such that G admits a k-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. Given a graph G, the problem of deciding whether G is k-colorable or not is called k-Colorability. Consider now a partition $\mathcal{V} = (V_1, V_2, \ldots, V_p)$ of the vertex set V of G. The sets $V_1, \ldots, V_p$ are called *clusters* and $\mathcal{V}$ is called a *clustering*. A *selection* is a subset of vertices $V' \subseteq V$ such that $|V' \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$. A *selective k-coloring* of G, also called *partition coloring* in the literature, with respect to $\mathcal{V}$ is defined by (V', c) where V' is a selection and c is a k-coloring of $G[V']$. We may define the following two problems:

* Corresponding author. Tel.: +90 212 359 66 76.
E-mail addresses: demange@essec.edu (M. Demange), tinaz.ekim@boun.edu.tr (T. Ekim), ries@lamsade.dauphine.fr (B. Ries), tanasescu@essec.edu (C. Tanasescu).

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SEL-COL
Input: An undirected graph $G = (V, E)$ and a clustering $\mathcal{V} = (V_1, \dots, V_p)$ of V.
Output: A selection V^* such that $\chi(G[V^*])$ is minimum.

Let $k \geqslant 1$ be a fixed integer.

k-DSEL-COL
Input: An undirected graph $G = (V, E)$ and a clustering $\mathcal{V} = (V_1, \dots, V_p)$ of V.
Question: Does there exist a selection V' such that $\chi(G[V']) \leqslant k$?

The smallest integer k for which a graph G admits a selective k-coloring with respect to $\mathcal{V}$ is called the *selective chromatic number* of G and is denoted by $\chi_{SEL}(G, \mathcal{V})$. It is obvious to see that $\chi_{SEL}(G, \mathcal{V}) \leqslant \chi(G)$ for every clustering $\mathcal{V}$ of V. For both problems above, adding or/and deleting edges between vertices of a same cluster does not affect the solution. However, when considering specific classes of graphs, adding or deleting edges in a cluster may not be allowed if the graph class we are considering is not stable under such graph operations. In this case, it is sometimes interesting to consider instances for which all clusters are cliques. We call this case *compact clustering* (Demange, Monnot, Petrica, & Ries, 2013).

The selective graph coloring problem has been considered by several authors. Some complexity results for restrictive classes of graphs can be found in (Demange et al., 2013; Erlebach & Jansen, 2001). In (Guangzhi & Simha, 2000), some constructive heuristics are proposed whereas a tabu search algorithm for SEL-COL is developed in (Noronha & Ribeiro, 2006). Concerning exact algorithms, a branch-and-cut algorithm and a branch-and-price algorithm are given along with computational results in respectively (Frota, Maculan, Noronha, & Ribeiro, 2010; Hoshino, Frota, & de Souza, 2011). More graph theoretical results concerning the selective graph coloring problem can be found in (Bonomo, Cornaz, Ekim, & Ries, 2013).

To the best of our knowledge (Guangzhi & Simha, 2000; Noronha & Ribeiro, 2006) are the only papers where the selective graph coloring problem is explicitly mentioned to model a real life problem which is called routing and wavelength assignment. However, we notice that the selective graph coloring problem has a huge potential to extend the use of the standard coloring problem to many other real life applications. The aim of this paper is to emphasize how several real life problems can be modeled using the selective graph coloring model whereas the usual graph coloring model would not be able to handle them. Section 2 presents examples such as routing and wavelength assignment, dichotomy-based constraint encoding, frequency assignment, timetabling, quality test scheduling, berth allocation and multiple stuck TSP. Each model motivates the study of SEL-COL in some specific class of graphs and rises in particular questions about complexity. These questions are studied in (Demange et al., 2014) and in this paper we will only mention the results without the proofs. We first start with a few remarks about the complexity of the selective graph coloring problem.

1.2. Some remarks on the complexity of the selective graph coloring problem

As mentioned above, a solution for SEL-COL consists in a selection V^* such that $\chi(G[V^*])$ is minimum. However, the value associated with V^*, namely $\chi(G[V^*])$, may be hard to determine even if V^* is known. In other words, when we can compute efficiently an optimal solution for SEL-COL for an instance $(G, \mathcal{V})$, this does not necessarily imply that its value $\chi_{SEL}(G, \mathcal{V})$ can be computed in polynomial time. More precisely, it is still NP-hard to compute the selective chromatic number in general even when an optimal selection V^* is known. For this reason, we will sometimes distinguish between the hardness of SEL-COL (selection process) and the hardness of deciding whether a given selection induces a k-colorable graph.

The following examples of reductions point out this difference. First note that, for any $k \geqslant 1$, k-DSEL-COL is in NP in general graphs. Indeed, given an instance $(G, \mathcal{V})$ with a set $V' \subseteq V$ and a coloring of $G[V']$, in order to test whether $\chi(G[V'])) \leqslant k$, it is enough to verify whether V' meets each cluster exactly once and the color assignment defines a k-coloring of the graph $G[V']$. This can clearly be done in polynomial time. Furthermore, 3-DSEL-COL generalizes the usual 3-coloring problem (i.e, given a graph $G = (V, E)$, we want to know if G admits a coloring of V using at most 3 colors) for which each vertex is a cluster on its own, and hence it is NP-complete. However determining an optimal selection is easy since there is only one solution: all vertices must be selected since each vertex represents a cluster.

Showing the NP-hardness of SEL-COL requires another reduction. Consider an instance $G = (V, E)$ of 3-COLORABILITY. Let $G' = (V', E')$ be a graph composed of 3 independent copies of G and define the following clustering $\mathcal{V}$ of V': each cluster contains exactly the 3 copies of a same vertex in G. Then, G is 3-colorable if and only if an (optimal) selection for the instance $(G', \mathcal{V})$ is a stable set (i.e., $\chi_{SEL}(G', \mathcal{V}) = 1$). Indeed, if there is a selection which is a stable set, then a 3-coloring of G can be found by coloring a vertex with color $i \in \{1, 2, 3\}$ if its ith copy in G' has been selected. Conversely, a 3-coloring of G allows to find a selection which is a stable set by choosing the ith copy of a vertex in G' if it is colored with color $i \in \{1, 2, 3\}$. Note that deciding whether a given selection is 1-colorable is trivially polynomial.

The two reductions above show that 3-DSEL-COL is NP-complete even with clusters of size 1 while 1-DSEL-COL (resp. SEL-COL) is NP-complete (resp. NP-hard) even if clusters are of size 3. However, in the first case the selection is trivial but evaluating the corresponding selective chromatic number is hard while in the second case finding the required selection is hard but it is trivial to decide whether the chromatic number of the graph induced by any selection is 1. *More generally, any hardness result for k-DSEL-COL, with $k \leqslant 2$, or for SEL-COL in graph classes for which graph coloring is polynomial, points out the hardness of the selection process.* Note that in many cases both steps – finding a selection and evaluating its value – can be hard.

Finally, note that the last reduction above can be adapted to show that SEL-COL even generalizes the k-List-coloring problem. Here, with every vertex v of the instance graph $G = (V, E)$, we associate a list $L(v) \subseteq \{1, \dots, k\}$ of allowed colors and the problem consists in deciding whether there is a k-coloring of G such that each vertex gets a color from its list. We construct a graph $G' = (V', E')$ as follows. For every vertex $v \in V$ and every color $c \in L(v)$ we define a vertex (v, c) of G'. We then link (v, c) and (v', c') if $c = c'$ and v, v' are linked in G. The clustering $\mathcal{V}$ of V' is obtained by putting all vertices $(v, c), c \in L(v)$ in a same cluster. Clearly, G admits a k-list coloring if and only if $\chi_{SEL}(G', \mathcal{V}) = 1$.

2. Some models and related classes of graphs

In this section, we present five different types of applications each of which motivates SEL-COL in a particular class of graphs. For each of these classes, we mention the related computational complexity for solving the problem.

2.1. Routing and wavelength assignment

Optical networks appear in a large number of applications including among others high performance computing and