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Discrete Optimization

Min–Max vs. Min–Sum Vehicle Routing: A worst-case analysis

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ABSTRACT

The classical objective function of the Vehicle Routing Problem (VRP) is to minimize the total distance traveled by all vehicles (Min–Sum). In several situations, such as disaster relief efforts, computer networks, and workload balance, the minimization of the longest route (Min–Max) is a better objective function. In this paper, we compare the optimal solution of several variants of the Min–Sum and the Min–Max VRP, from the worst-case point of view. Our aim is two-fold. First, we seek to motivate the design of heuristic, metaheuristic, and matheuristic algorithms for the Min–Max VRP, as even the optimal solution of the classical Min–Sum VRP can be very poor if used to solve the Min–Max VRP. Second, we aim to show that the Min–Max approach should be adopted only when it is well-justified, because the corresponding total distance can be very large with respect to the one obtained by optimally solving the classical Min–Sum VRP.

1. Introduction

The *Vehicle Routing Problem* (VRP) is the problem of determining a set of routes that visit a set of customers at minimum distance, where each route satisfies a capacity constraint. This problem is interesting both from the theoretical and the practical points of view. In fact, finding an optimal solution is really challenging and this problem is solved daily by companies worldwide. In the last 50 years, numerous variants have been studied, including the case with one vehicle (*Traveling Salesman Problem* – TSP), the case with multiple uncapacitated vehicles (*Multiple TSP* – MTSP), and the more traditional case with several capacitated vehicles (*Capacitated VRP* – CVRP). The latter case, which was introduced in Dantzig and Ramser (1959), plays a central role in distribution management. The first exact algorithms for the CVRP were proposed by Christofides, Mingozzi, and Toth (1981a, 1981b). The best known exact algorithms are the ones proposed by Fukasawa et al. (2006) and Baldacci, Christofides, and Mingozzi (2008), Baldacci, Mingozzi, and Roberti (2011). For a recent survey on exact algorithms for the CVRP, we refer the reader to Baldacci, Mingozzi, and Roberti (2012). Although it is possible to optimally solve instances of the TSP with several thousands of customers, the CVRP remains very difficult to solve optimally, even if a few hundred customers are considered. Therefore, both heuristic and metaheuristic algorithms have been proposed for its solution. The most famous heuristic is the Savings algorithm by Clarke and Wright (1964). The best known metaheuristics are the Adaptive large neighborhood search by Pisinger and Ropke (2007) and the Hybrid genetic algorithm recently proposed by Vidal, Crainic, Gendreau, and Prins (2014). We refer to Toth and Vigo (2002) and Golden, Raghavan, and Wasil (2008) for two comprehensive books on the VRP and to Laporte, Toth, and Vigo (2013) for a recent overview of exact, heuristic, and metaheuristic approaches.

The classical objective function of the VRP is the minimization of the total distance traveled by all vehicles (Min–Sum). In this paper, we also focus on the case in which the aim is to minimize the longest route (Min–Max). This new objective function is important in several situations. For example, in disaster relief efforts the aim is to serve all victims as soon as possible, in computer networks the aim is to minimize the maximum latency between a server and a client, in workload balance the aim is to balance the amount of work among drivers on a given day or across a time horizon. A limited number of papers is devoted to the Min–Max VRP. A tabu search algorithm is proposed in França, Gendreau, Laporte, and Müller (1995) for the Multiple TSP. Averbakh and Berman (1996) study the problem in which two salesmen must visit nodes on a tree. Applegate, Cook, Dash, and Rohe (2002) develop specialized cutting planes and a distributed search algorithm to solve the so-called *Newspaper routing problem*. Carlsson, Ge, Subramaniam, and Ye (2009) study the multi-depot case and propose an LP-based balancing approach and a region partition

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heuristic. Wang, Golden, and Wasil (2013) develop a three-phase heuristic able to significantly improve upon the LP-based balancing approach. Ren (2011) proposes a hybrid genetic algorithm. Yakici and Karasakal (2013) propose a heuristic approach for the case with split deliveries and heterogeneous demands. Campbell, Vandenbussche, and Hermann (2008) propose for the first time a comparison of the solutions obtained with alternative objective functions. They define Min–Max and Min–Sum in a different way: Min–Max aims at minimizing the arrival time to the latest customer and Min–Avg (or Min–Sum) aims at minimizing the average arrival time or, equivalently, the sum of the arrival times to the customers. The paper by Huang, Smilowitz, and Balcik (2012) extends this work by studying how alternative objectives, based on equity, efficiency, and efficacy metrics, influence the structure of the routes. The solutions obtained on the basis of the definitions of Min–Max and Min–Sum used in Campbell et al. (2008) can be very different from the ones obtained on the basis of our definition of Min–Max and Min–Sum. Consider for example the simpler TSP case. Our Min–Sum and Min–Max objectives are equivalent, while if the aim is to minimize the latest arrival, the routing can significantly differ. In fact, Campbell et al. (2008) show that the worst-case ratio between the total length of the route obtained by minimizing the latest arrival and the total length of the route obtained by minimizing the total length is 3/2.

The length of the longest route in the Min–Sum VRP is not lower than the length of the longest route in the Min–Max VRP, while the total distance in the Min–Max VRP is not lower than the total distance in the Min–Sum VRP. Our aim is to consider several variants of the VRP. For each of these variants, we aim to answer the following questions:

1. What is the ratio of the length of the longest route in the Min–Sum VRP to the length of the longest route in the Min–Max VRP, in the worst case?
2. What is the ratio of the total distance of the Min–Max VRP to the total distance of the Min–Sum VRP, in the worst case?

The answer to the first question tells us if minimizing the total distance can imply a significant increase in the length of the longest route. In that case, the design of heuristic, metaheuristic, and matheuristic algorithms for the Min–Max VRP is well-motivated. The answer to the second question tells us if minimizing the longest route can imply a significant increase in the total distance. In that case, this objective should be really well-justified to be adopted.

The remainder of the paper is organized as follows. In Section 2, the variants of the VRP studied in the paper are formally described. In Section 3, the worst-case analysis concerning the Capacitated VRP with an infinite number of vehicles is shown. Section 4 focuses on the Capacitated VRP with a finite number of vehicles. Section 5 concerns the Multiple TSP. Section 6 focuses on the Service time VRP with a finite number of vehicles. Some conclusions are presented in Section 7.

2. Description of problems

Let $G(V,E)$ be a complete graph, where $V = \{0, 1, \ldots, n\}$ is the set of vertices and E is the corresponding set of edges. Vertex 0 corresponds to the depot, while vertices $1, 2, \ldots, n$ correspond to the customers. Each customer has to be served in full by one route (i.e., splitting of the demand is not allowed). Let c_{ij} be the distance corresponding to the edge $(i,j) \in E$. We consider the following four variants of the VRP:

1. *Capacitated VRP with an infinite number of vehicles:* Each customer $i = 1, 2, \ldots, n$ has a demand $d_i > 0$ not greater than the vehicle capacity C. An infinite fleet of vehicles is available.
2. *Capacitated VRP with a finite number of vehicles:* Each customer $i = 1, 2, \ldots, n$ has a demand $d_i > 0$ not greater than the vehicle capacity C. At most k vehicles are available.
3. *Multiple TSP:* The customers just have to be visited (i.e., no demand has to be satisfied). Each vehicle has infinite capacity. Exactly k routes have to be determined.
4. *Service time VRP with a finite number of vehicles:* Distances are replaced by travel times. Customer demands are given in terms of service times. The duration of any route is the sum of travel time and service times of the customers visited by the route. At most k vehicles are available and there is no limit on the total load or duration of a route.

In the Min–Sum VRP, the problem is to determine a set of routes that minimizes the total distance (or total time). Instead, in the Min–Max VRP, the problem is to determine a set of routes that minimizes the length (or duration) of the longest route.

3. Capacitated VRP with an infinite number of vehicles

In the Capacitated VRP with an infinite number of vehicles, each customer $i = 1, 2, \ldots, n$ has a demand $d_i > 0$ not greater than the vehicle capacity C. An infinite fleet of vehicles is available.

Let us denote by r^{∞}_{MM} the length of the longest route in the optimal solution of the Min–Max Capacitated VRP with an infinite number of vehicles and by r^{∞}_{MS} the length of the longest route in the optimal solution of the Min–Sum Capacitated VRP with an infinite number of vehicles.

Theorem 1. *There exists an instance class with parameter ϵ such that $\frac{r^{\infty}_{MS}}{r^{\infty}_{MM}} \to \infty$ for $\epsilon \to \infty$.*

Proof. Let $0 \leqslant \epsilon < 1$ be a real number such that $\frac{1}{\epsilon}$ is an integer. Consider the following instance class with parameter ϵ:

Example 1.

- Single depot called node 0.
- Number of customers: $n = 1 + \frac{1}{\epsilon}$ (nodes $1, 2, \ldots, 1 + \frac{1}{\epsilon}$).
- Vehicle capacity: $C = \frac{1}{\epsilon}$.
- Demand of customer 1: $d_1 = \frac{1}{\epsilon}$.
- Demand of customers $i = 2, 3, \ldots, n : d_i = \epsilon$.
- Depot to customer distances: $c_{0i} = 1$, for $i = 1, 2, \ldots, n$.
- Customer to customer distances: $c_{ij} = 1 - \epsilon$ for $i,j = 1, 2, \ldots, n$, $i \neq j$.

The corresponding optimal solutions are shown in Fig. 1(a) and (b).

An optimal solution of the Min–Sum Capacitated VRP with an infinite number of vehicles is the following: Serve customer 1 directly and all the remaining customers $2, 3, \ldots, n$ in the same route. In fact, since $d_1 = C$ and splitting of the demand is not allowed, customer 1 has to be served directly. Moreover, the length of the route serving customers $2, 3, \ldots, n$ is $1 + (1 - \epsilon)(\frac{1}{\epsilon} - 1) + 1 = \frac{1}{\epsilon} + \epsilon$. This length cannot be reduced by using more routes to serve these customers. In fact, let $2 \leqslant R \leqslant \frac{1}{\epsilon}$ (R integer) be the number of routes to serve these customers. The corresponding length is $2R + (1 - \epsilon)(\frac{1}{\epsilon} - R)$, which is greater than $\frac{1}{\epsilon} + \epsilon$ for $R \geqslant 2$. Therefore, the length of the longest route is $r^{\infty}_{MS} = \frac{1}{\epsilon} + \epsilon$.

An optimal solution of the Min–Max Capacitated VRP with an infinite number of vehicles is the following: Serve each customer $1, 2, \ldots, n$ directly. In fact, since customer 1 has to be served