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An approximate moving boundary method for American option pricing

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ABSTRACT

We present a method to solve the free-boundary problem that arises in the pricing of classical American options. Such free-boundary problems arise when one attempts to solve optimal-stopping problems set in continuous time. American option pricing is one of the most popular optimal-stopping problems considered in literature. The method presented in this paper primarily shows how one can leverage on a one factor approximation and the moving boundary approach to construct a solution mechanism. The result is an algorithm that has superior runtimes-accuracy balance to other computational methods that are available to solve the free-boundary problems. Exhaustive comparisons to other pricing methods are provided. We also discuss a variant of the proposed algorithm that allows for the computation of only one option price rather than the entire price function, when the requirement is such.

1. Introduction

This paper presents a fast and accurate method to price an American put option written on one underlying asset. An American put option is a contract written on an underlying asset and gives the holder the right to sell the asset for a pre-specified price (strike) on or before a pre-specified date (maturity). An American call, on the other hand, provides the holder the right to buy the underlying asset. Unlike European options where the holder can exercise the option only on the maturity date, the possibility of early exercise makes the pricing of American options a problem in stochastic optimization. While a closed-form solution for the price of European options is derived in the seminal Black and Scholes (1973) paper, there exists no analogous result for American options.

American option pricing belongs to a very common class of problems called *Optimal-stopping problems*. The American option pricing problem is very often used as the canonical optimal-stopping problem to construct and demonstrate new computational methods that can readily be adapted to other optimal-stopping problems. McKean (1965) and Merton (1973) show that the price of an American option satisfies a partial differential equation (PDE) with boundary conditions governed by a boundary that is not known *a priori* and needs to be computed as part of the solution itself. Such problems are called free-boundary problems. Due to the lack of closed-form solutions for the price, researchers and practitioners have had to rely on numerical and approximation schemes to price these options. These methods fall broadly into three categories, those which use numerical techniques, those which use analytical approximations of the price, and those which use Monte Carlo simulation.

Some of the first methods developed to tackle the problem of pricing an American option use numerical techniques. Brennan and Schwartz (1977) use a finite difference scheme to transform the PDE into a system of linear equations. Solving this system recursively provides the option price for all times and underlying asset prices, or the price function. The Projected Successive Over-relaxation (PSOR) technique proposed by Cryer (1971) has also been used to price American options numerically. Dempster and Hutton (1999) use a finite difference approximation and derive a linear programming (LP) problem at each time step. Each of these LP problems is solved using the simplex method. The authors conclude that the simplex method is roughly comparable to the PSOR technique. A front-fixing method is proposed in Wu and Kwok (1997) and Nielsen, Skavhaug, and Tveito (2002) to compute option prices. The front-fixing method utilizes a change in variables to transform the free-boundary problem into a nonlinear problem on a fixed domain. Nielsen et al. (2002) also propose a penalty method to price American put options, where the unknown boundary is removed by adding a penalty term, again leading to a nonlinear problem posed on a fixed domain. More recently, Muthuraman (2008) uses a moving boundary approach to convert the free-boundary problem into a sequence of fixed-boundary problems which are easier to solve and can be solved

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using standard PDE-solvers. Using a guess of the exercise policy (the unknown boundary), one obtains a fixed-boundary problem. The price function obtained by solving this problem is used to update the exercise policy, and the process is repeated till convergence. These methods compute the price function for a particular option, after which the option price for a particular underlying asset price at a particular time to expiration is read off from this function. A variety of methods have been proposed to obtain option prices for a single underlying asset price at a single time to expiration. Cox, Ross, and Rubinstein (1979) develop the binomial-tree method which is widely in use today. Variants include the trinomial-tree method (Boyle (1986)). Some methods use the integral representations of the optimal exercise boundary and the price function developed independently in Carr, Jarrow, and Myneni (1992), Jacka (1991) and Kim (1990) to price the option. Broadie and Detemple (1996) derive upper and lower bounds for the price of an option using capped call options and the integral representation for call options. As the integral representation of the price function is dependent on the unknown optimal exercise boundary, these methods first have to determine the boundary recursively, and then price the option. Huang, Subrahmanyam, and Yu (1996) is an example of such a method. Others, such as Omberg (1987) and Chesney (1989), use exponential approximations to represent the boundary, and price the option using these boundaries, but with limited results. Ju (1998) considers a multi-piece exponential formula to represent the boundary and obtains a closed-form solution for the price of the option based on this representation. Approximating the boundary, however, requires the calculation of twelve parameters iteratively while pricing the option involves the calculation of complex integrals.

The second category of methods use analytical approximations to represent the price of an option. MacMillan (1986) and Barone-Adesi and Whaley (1987) develop quadratic approximations for the option price. These methods are not convergent, and have trouble pricing long-maturity options accurately. To correct this problem, Ju and Zhong (1999) develop an approximation based on the method proposed in Barone-Adesi and Whaley (1987). While this improved method prices long-maturity options more accurately, it is still not convergent. Geske and Johnson (1984) view an American option as a sequence of Bermudan options and propose an approximation of the option price consisting of an infinite sequence of cumulative normal functions. Bunch and Johnson (1992) propose a modified two-point Geske–Johnson approach. Carr and Faguet (1996) view put options as the limit to a sequence of perpetual option values which are subject to default risk, and use this view to derive approximations for the price of an option. More recently, Zhu (2006) derives a semi-closed form solution for the price of the option as a Taylor series expansion consisting of infinite terms, but requiring thirty terms for an accurate option price.

Monte Carlo simulation techniques have gained popularity as a result of their ability to price options written on several underlying assets considerably quicker than numerical methods. Boyle (1977) first proposed the use of simulation in option pricing, with Tilley (1993) developing the first computational scheme capable of implementing simulation techniques. Broadie and Glasserman (1997), Carriere (1996), Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (1999) and Ibáñez and Zapatero (2004) are other examples of option pricing methods using Monte Carlo simulation.

1.1. Contributions

In this paper, we construct a numerical scheme that quickly computes the price function for an American put option by determining an approximate optimal exercise boundary. As highlighted by others, such as Ju (1998) and Glasserman (2004), an extremely accurate estimate of the optimal exercise boundary is not necessary to obtain accurate option prices. The method proposed here is a variant of the moving boundary approach presented in Muthuraman (2008). The key difference here is that by leveraging on a one factor approximation, we are able to obtain an algorithm that has significantly better runtime-error balance. Moreover, the primary disadvantage of the moving boundary approach is that, when the requirement is such, it cannot be readily modified to efficiently compute only one option price without having to compute the entire price function. The variant proposed herein, however, while taking advantage of the efficiency of the moving approach, does allow for adaptation into an efficient algorithm for computing a single option price. In this paper, we will restrict our focus to only American put options since pricing call options is conceptually the same with identical equations (but for the boundary condition). Furthermore, the price of a call option written on the same underlying asset can be calculated using put-call parity relations (see, for e.g. McDonald & Schroder (1998)) once the price of the put option is determined. Most existing methods just provide the price of the option, with the optimal exercise boundary having to be calculated as a post-processing step. Knowledge of the boundary allows an investor to make optimal decisions without having to calculate the theoretical option price each time a decision has to be made. The investor has just to compare the current asset price with the boundary value at that time in order to act optimally. A method which provides both a price and the boundary quickly and efficiently is thus highly beneficial and desirable.

The central idea in the paper is that when a parameterized approximation to the boundary exists, one can use the methodology proposed in the paper to create a very efficient pricing solution. Though we demonstrate this using the classical American option pricing setting, the benefit of such an approach increases significantly when dealing with multidimensional cases that arise in the presence of stochastic interest rates or multiple assets. Of course, one has to find a good parameterized approximation of the boundary and also ensure convergence. However as demonstrated here, even in the simplest one dimensional case the benefit of such an approach is significant.

The structure of the paper is as follows. In Section 2, the problem of pricing an American put option is laid out theoretically. Section 3 discusses in detail the numerical scheme highlighted above. We provide a test case in Section 5 to further illustrate the mechanism behind the method, together with exhaustive numerical results. In Section 6, we propose a modification of the method to calculate the option price for a single underlying asset price and time to expiration. Section 7 summarizes this work and provides directions for future research in this area.

2. The American option pricing problem

As in the Black–Scholes setting, we assume a perfect market (no transaction costs, market is complete and arbitrage-free) with

$$dX(t) = (r - \delta)X(t)dt + \sigma X(t)dW(t)$$

representing the dynamics of the price process of the underlying asset, where $r > 0$ represents the risk-free rate of interest, $\delta \geqslant 0$ represents the continuous dividend yield, $\sigma > 0$ represents the volatility of the underlying asset and $W(t)$ is a standard one-dimensional Brownian motion.

Though we would like to investigate the effect of non-zero dividends in the numerical section, especially because of the need to compare against benchmark cases, the inclusion of dividends is straightforward. Hence for notational simplicity, we ignore dividends in the rest of the paper, though we will included cases with