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Continuous Optimization

# Exploring trust region method for the solution of logit-based stochastic user equilibrium problem

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ABSTRACT

In this research paper, we explored using the trust region method to solve the logit-based SUE problem. We proposed a modified trust region Newton (MTRN) algorithm for this problem. When solving the trust region SUE subproblem, we showed that applying the well-known Steihaug-Toint method is inappropriate, since it may make the convergence rate of the major iteration very slow in the early stage of the computation. To overcome this drawback, a modified Steihaug-Toint method was proposed. We proved the convergence of our MTRN algorithm and showed its convergence rate is superlinear.

For the implication of our algorithm, we proposed an important principle on how to select the basic route for each OD pair. We indicated that it is a crucial principle to accelerate the convergence rate of the minor iteration (i.e. trust region subproblem-solving iteration). In this study, other implication issues for the SUE problem are also considered, including the computation of the trial step and the strategy to ensure strict feasibility iteration point. We compared the MTRN algorithm with the Gradient Projection (GP) algorithm on the Sioux Falls network. Some results of numerical analysis are also reported.



## 1. Introduction

There has been considerable interest in recent years in stochastic user equilibrium (SUE) problem for traffic assignment. In the literature, SUE was proposed to relax the unrealistic assumption behind the user equilibrium (UE) problem, which states that people have perfect knowledge of network conditions. Daganzo and Sheffi (1977) first defined SUE principle. They assume that people have different perception errors when selecting routes, and at the SUE equilibrium, no traveler believes he can improve his travel time by unilaterally changing routes.

The most widely studied SUE models are the logit model and the probit model. The logit model assumes that people's perception error follows a logistic distribution (Akamatsu, 1996; Bell, 1995; Chen & Alfa, 1991; Dial, 1971; Fisk, 1980; Leurent, 1997). The probit model assumes that people's perception error has a normal distribution (Daganzo & Sheffi, 1977; Sheffi & Powell, 1982). Theoretically, each model have advantages and disadvantages. Logit model has well-known weaknesses such as their inability to take proper account of overlapping or correlated paths. However, it has an advantage of analytical simplicity. Probit model is more behaviorally appealing than logit model, but suffers the disadvantage either of requiring Monte Carlo techniques or of complete path enumeration. In the literature, the logit model has enjoyed much greater attention than the probit model. It is widely used not only for theoretical investigation (Ghatee & Hashemi, 2009; Guo, Yang, & Liu, 2010; Zhou, Chen, & Wong, 2009), but also for practical implementation (Aros-Vera, Marianov, & Mitchell, 2012; García-Ródenas & Marín, 2009; Haase & Müller, 2014). In this study, we will concentrate on the logit model.

Generally speaking, solution algorithms for the logit based SUE problem can be divided into two classes: link-based algorithms and path-based algorithms. Link-based algorithms do not need explicit path enumeration. They only assume an implicit path choice set, such as the use of all efficient paths (Dial, 2001; Maher, 1998), or all cyclic and acyclic paths (Akamatsu, 1996; Bell, 1995). On the other hand, path-based algorithms require explicit choosing a subset of feasible paths prior to the assignment. Therefore, a large variety of methods can be used to generate a more realistic path choice set. For different types of path choice set generation methods, we can refer to Azevedo, Santos Costa, Silvestre Madera, and Vieira Martins (1993), Bekhor, Ben-Akiva, and Ramming (2006), Ben-Akiva, Bergman, Daly, and Ramaswamy (1984), Cascetta, Nuzzolo, Russo, and Vitetta (1996), and De la Barra, Perez, and Anez (1993)

This paper concentrates on the path-based algorithms for the logit SUE model. Bekhor and Toledo (2005) proposed using the

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Gradient Projection (GP) method (Bertsekas, 1999) to solve this problem. In their study, the gradient of the objective function is projected on a linear manifold of the equality constraints, with the scaling matrix being diagonal elements of the Hessian. Note that algorithm GP still retains a linear convergence rate, it could be slow as it is approaching the optimal solution. Hence, we wonder to know whether methods that enjoy superlinear convergence rate (such as the Newton type method) can further improve the computation efficiency. At first sight, this is obvious, since faster convergence rate undoubtedly results in less computation time. However, it is not the case. As we know, superlinear convergence rate is a local property, only if the iteration point lies within a neighborhood of the optimal solution can we obtain such rate of convergence. But we do not know the optimal SUE solution a priori, hence we cannot find an initial point that is near the solution before finishing the computation. Practical routine for choosing the initial point for the SUE problem is to obtain a logit type route flow pattern using free-flow travel times. Therefore, when this initial point is far from the optimal solution, the computation time may be high, and most of the computing time is consumed in the early stages of the iteration, at which the iteration point is far from the optimal solution.

As we know, trust region method is an important and efficient method in non-linear optimization. In this study, we will propose a modified trust region Newton (MTRN) algorithm to solve the logit-based SUE problem. Logit-based SUE problem is in essence a linear equality constraint optimization problem.We will first use the variable reduction method to transform it into an unconstrained one, and then use the trust region Newton method to solve it. However, when solving the trust region SUE subproblem, the well-known Steihaug-Toint method may be sometimes inappropriate. The reason is that in the early stage of iteration, the trust region radius should be adjusted so that the next iteration point satisfies the non-negative constraint. Hence it is usually very small. For small trust region radius, the Steihaug-Toint method usually terminates at the first iteration, which makes the trial step it generates only along the Cauchy direction (i.e., the steepest descent direction). Therefore, in the early iteration stage, sometimes the Steihaug-Toint method may be very slow and should not be used. Alternatively, a modified Steihaug-Toint method is proposed. This method does not suffer from the disadvantage that Steihaug-Toint method retains. It is a very efficient method for the SUE problem.

When applying the modified trust region Newton (MTRN) algorithm to the SUE problem, several practical issues are studied in our research, including the computation of the trial step, the strategy to ensure strict feasibility iteration point, and the principle of the choice of the basic route. These issues are very important to generate a fast and robust solution algorithm for the SUE problem.

The rest of this paper is structured as follows: In Section 2, we briefly outline trust region Steihaug-Toint (TRST) algorithm for the linear equality constraint optimization problem. In Section 3, we propose the modified trust region Newton algorithm (MTRN) for logit-based SUE problem. In Section 4, the MTRN algorithm and the GP algorithm are tested and compared on the Sioux Falls network. In Section 4, we provide conclusions and suggestions for future work.

## 2. A trust region Steihaug-Toint algorithm for the linear equality constraint optimization problem

### 2.1. Trust region method for major iteration

Consider the problem in the form:

$$\text{[P1]} \quad \text{minimize } f(x) \tag{1}$$

$$\text{subject to } Ax = b \tag{2}$$

where $f$ is twice continuously differentiable, and $A$ is an $m \times n$ matrix of full row rank. We further assume for convenience that $f$ is strictly convex, which guarantees that there is a global minimizer of [P1].

Let $\bar{x}_0$ be any feasible point for [P1], and $Z$ be an $n \times (n-m)$ basis matrix for the null space of $A$. It is well known that any other feasible point can be expressed as

$$x = \bar{x}_0 + Zy \tag{3}$$

where $y$ is an $(n-m)$-dimensional vector.

Therefore, [P1] is equivalent to the following unconstrained problem:

$$\text{[P2]} \quad \underset{y \in \Re^{n-m}}{\text{minimize}} \quad \phi(y) = f(\bar{x}_0 + Zy) \tag{4}$$

In this paper, we call $f(x)$ the original objective function. Its argument $x$ is the original variable. Correspondingly, $\phi(y)$ and its argument $y$ are called the reduced objective function and reduced variable, respectively.

If $g \triangleq \nabla f(x)$ and $H \triangleq \nabla^2 f(x)$ are the gradient and the Hessian matrix of $f$, we can define the reduced gradient and the reduced Hessian matrix of $f$ by the following expression:

$$\tilde{g} \triangleq \tilde{g}(y) = \nabla\phi(y) = Z^T f(x) \tag{5}$$

$$\widetilde{H} \triangleq \widetilde{H}(y) = \nabla^2\phi(y) = Z^T \nabla^2 f(x) Z \tag{6}$$

By assumption, the objective function $f(x)$ is strictly convex, then its Hessian matrix $H$ is positive definite. For any non-zero vector $y$, $(Zy)^T H(Zy) > 0$, implies $y^T(ZHZ)y > 0$. Therefore, the reduced Hessian matrix $\widetilde{H} = ZHZ$ is also positive definite. As a result, we conclude that the reduced objective function $\phi(y)$ is strictly convex, and [P2] is also a convex optimization problem.

We now briefly describe the trust region Steihaug-Toint (TRST) algorithm to solve [P2]. This algorithm consists of two phases. The major iteration phase applies the trust region framework to [P2], and creates a trust region subproblem. The minor iteration phase uses the Steihaug-Toint method to solve the subproblem approximately.

Let $y_k$ be the current iteration point, we first define a region around $y_k$:

$$\mathrm{B}_k = \{ y \in \Re^{n-m} | \|y - y_k\|_k \leqslant \Delta_k \} \tag{7}$$

where $\Delta_k$ is the trust region radius, and $\|\cdot\|_k$ is an iteration dependent norm.

Then we build a quadratic model that approximate the objective function $\phi(y)$ around $y_k$, and choose a step to be the approximate minimizer of the model within the trust-region. That is, we seek a solution of the following trust region subproblem:

$$\underset{p \in \Re^{n-m}}{\text{minimize}} \quad m_k(p) = \phi(y_k) + \tilde{g}_k^T p + \frac{1}{2} p^T \widetilde{H}_k p \tag{8}$$

$$\text{subject to} \quad \|p\|_{M_k} \leqslant \Delta_k \tag{9}$$

where

$$\|p\|_{M_k} = \sqrt{p^T M_k p} \tag{10}$$

and $M_k$ is a symmetric positive definite matrix that depend on iteration number $k$.

Define the ratio

$$\rho_k = \frac{\phi(y_k) - \phi(y_k + p_k)}{m_k(0) - m_k(p_k)} \tag{11}$$

In (11), the numerator is called the actual reduction, and the denominator is called the predicted reduction. This ratio measures the agreement between the actual reduction in $\phi$ and the predicted reduction in $m_k$.