European Journal of Operational Research 234 (2014) 392–401

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Mean–variance–skewness efficient surfaces, Stein's lemma and the multivariate extended skew-Student distribution

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ARTICLE INFO

Article history:
Received 5 October 2012
Accepted 10 July 2013
Available online 20 July 2013

Keywords:
Finance
Multivariate statistics
Utility theory

ABSTRACT

Recent advances in Stein's lemma imply that under elliptically symmetric distributions all rational investors will select a portfolio which lies on Markowitz' mean–variance efficient frontier. This paper describes extensions to Stein's lemma for the case when a random vector has the multivariate extended skew-Student distribution. Under this distribution, rational investors will select a portfolio which lies on a single mean–variance–skewness efficient hyper-surface. The same hyper-surface arises under a broad class of models in which returns are defined by the convolution of a multivariate elliptically symmetric distribution and a multivariate distribution of non-negative random variables. Efficient portfolios on the efficient surface may be computed using quadratic programming.

1. Introduction

If $\mathbf{X}$ is a random vector which has a full rank multivariate normal distribution $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $h(\cdot)$ is a scalar valued function of $\mathbf{X}$ which satisfies certain regularity conditions, then $\text{cov}\{\mathbf{X}, h(\mathbf{X})\} = \boldsymbol{\Sigma}E\{\nabla h(X)\}$, where $\nabla h(\mathbf{X})$ is the vector first derivatives of $h(\cdot)$ with respect to the elements of $\mathbf{X}$. This result is the multivariate generalisation of Stein's lemma, Stein (1973, 1981) and is reported in Liu (1994). Liu also points out that similar results hold for other distributions and gives some examples. In particular, he provides an expression for $\text{cov}\{X, h(X)\}$ for the case where the scalar random variable X has Student's t distribution. Landsman (2006) and Landsman and Nešlehová (2008) extend these results, with the latter paper providing an extension of Stein's lemma for multivariate elliptically symmetric distributions. (For general background see Fang, Kotz, & Ng, 1990.) This is an important class of distributions for portfolio theory. If the multivariate distribution of asset returns is member of the elliptically symmetric class, then the distributions of returns on a portfolio of the assets, which is an affine transformation of the vector of returns, is a member of the same class. Of arguably greater importance is the implication of Landsman and Nešlehova's extension of Stein's lemma. For portfolio selection $h(\mathbf{X}) = U(\boldsymbol{w}^T\mathbf{X})$, where $U(\cdot)$ is a utility function and w is the vector of portfolio weights. Under elliptical symmetry, and subject only to regularity conditions, the portfolios of expected utility maximisers will be located on Markowitz' mean–variance efficient frontier. Since Markowitz' original paper, mean–variance portfolio selection has been the subject of great interest and many articles, books and monographs. The method has been the subject of both praise and criticism. Nonetheless, the fact that the efficient frontier arises under conditions which are far more general than quadratic utility or normally distributed returns is one of many the reasons for its longevity and a tribute to the robustness of the original theory.

As well as non-normality, almost always in the form of fat tails, it has long been accepted that returns on some financial assets are not symmetrically distributed. Indeed, there is a long literature about skewness in asset returns. This dates back at least to the foundation papers of Samuelson (1970), Arditti and Levy (1975) and Kraus and Litzenberger (1976). In addition to these and other theoretical papers, there are numerous articles which report empirical studies of skewness. There are also numerous papers which are concerned with portfolio selection in the presence of skewness. Well known works include, but are not limited to, papers by Chunhachinda, Dandapani, Hamid, and Prakash (1997), Sun and Yan (2003), de Athayde and Flôres (2004), Briec, Kerstens, and Jokung (2007), Li, Qin, and Kar (2010), Goh, Lim, Sim, and Zhang (2012) and Matmoura and Penev (2013). Many of these have been summarised recently in the review paper by Adcock, Eling, and Loperfido (2012). Some papers (Jondeau & Rockinger, 2006, for example) employ Taylor series methods to justify the inclusion of a cubic term in an approximation to a utility function. It is natural therefore to enquire to what extent there is a mean–variance–skewness extension of Markowitz' efficient frontier; that is, a single mean–variance–skewness surface on which the portfolios of all expected utility maximisers will be located.

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0377-2217/$ - see front matter
http://dx.doi.org/10.1016/j.ejor.2013.07.011

The view taken in this paper is that the starting point for the development of portfolio selection theory is a coherent multivariate probability distribution for asset returns. In principle, this allows the computation of expected utility and thus makes explicit the nature of the relationship between a set of efficient portfolio weights, the parameters of the distribution and the utility function itself. For symmetrically distributed returns, members of the elliptically symmetric class are attractive because they are tractable with respect to the requirements of efficient portfolio selection. The same requirements imply that skewed multivariate distributions should also be selected for portfolio selection, at least in part, because of their tractability. A well-known skewed multivariate distribution which meets the requirement of tractability is the multivariate skew-normal distribution (MSN henceforth). This was introduced in its original form by Azzalini and Dalla-Valle (1996). The derivation considers a random vector of length $(n+1)$, $(\mathbf{U}^T, V)$, $\mathbf{U}$ an n-vector and V a scalar, which has a multivariate normal distribution with mean vector $(\boldsymbol{\mu}^T,0)$ and full rank covariance matrix which is arbitrary except for the diagonal entry corresponding to V, which is unity. The multivariate skew-normal arises as the conditional distribution of $\mathbf{U}$ given that $V > 0$. The MSN distribution was also reported as a conditional distribution in Azzalini and Capitanio (1999).

A modified version of this distribution, which has an additional parameter, was reported by Arnold and Beaver (2000). For this version of the distribution, before truncation the mean of the scalar variable V is arbitrary. This distribution was also reported independently in Adcock and Shutes (2001), who were the first to employ it in finance. This modification, which is generally known as the multivariate extended skew-normal (MESN) distribution, is attractive for applications in finance. The additional parameter offers more flexibility in modelling higher moments than the MSN. Explicit formulae for the moments are in Adcock and Shutes (2001).

From the perspective of portfolio selection the MESN distribution also admits an extension to Stein's lemma. Adcock (2007) shows that under the MESN distribution there is a single mean–variance–skewness efficient surface. As is well known, the derivation of both the MSN and MESN distributions employs a single unobserved or hidden variable which is non-negative. From a finance perspective, this implies that there is a single source of asymmetry in returns; a one factor model for skewness. A more complex skew-normal model in which there is more than one hidden non-negative variable is described in Sahu, Dey, and Branco (2003), Gonzàlez-Farías, Dominguez-Molína, and Gupta (2004) and Arellano-Valle and Azzalini (2006). This is generally known as the closed skew-normal (CSN) distribution. Adcock (2007) describes an extended version of this distribution, denoted CESN, and shows that there is an extension of Stein's lemma for it. Under this distribution, skewness is generated by multiple factors and there is a single mean–variance–skewness hyper-surface for expected utility maximisers, with a dimension for each skewness factor. The CSN distribution is employed in portfolio selection by Harvey, Liechty, Liechty, and Müller (2010), who extend Sahu et al.'s (2003) model and thus deal with co-skewness between assets.

Both the MESN and CESN distributions are tractable from the perspective of portfolio theory. However, even though the additional parameter(s) give flexibility in modelling moments, it is debatable whether these distributions will always adequately deal with the fat tails effects observed in empirical data. The first objective of this paper, therefore, is to present results for portfolio selection based on multivariate skew-Student-t distributions. These are increasingly well-known extensions of the multivariate skew-normal. Initial papers by Branco and Dey (2001) and Azzalini and Capitanio (2003) have been followed by articles by several authors including Azzalini and Genton (2008) and Arellano-Valle, Branco, and Genton (2006). A multivariate extended skew-Student-t, MEST, distribution and its properties are described in Adcock (2002, 2010) and Arellano-Valle and Genton (2010). Like its skew-normal counterpart, the MEST distribution employs one truncated variable.

The notable paper by Sahu et al. (2003), already referred to above, presents a general multivariate skew-elliptical distribution. This assumes that a $2n$-vector $(\mathbf{U}^T, \mathbf{V}^T)$ has a multivariate elliptically symmetric distribution and then conditions on all elements of $\mathbf{U}$ being positive. The expected values of the elements of $\mathbf{U}$ are assumed to be equal to zero. Their general results are exemplified using the skew-normal and skew-Student cases. Arellano-Valle and Genton (2010) also describe a version of the skew-Student t distribution, CEST henceforth, which has more than one truncated variable. In this case, the expected values of the elements of $\mathbf{U}$ are not restricted. In their paper, the CEST distribution extends related earlier work by Arellano-Valle and Azzalini (2006).

The second objective of this paper is to present corresponding results for multivariate models of asset returns which are members of the class defined by Simaan (1987, 1993). He proposes that the n-vector of returns on financial assets should be represented as $\mathbf{X} = \mathbf{U} + \lambda V$. The n-vector $\mathbf{U}$ has a multivariate elliptically symmetric distribution and is independent of the non-negative univariate random variable V, which has an unspecified skewed distribution. The n-vector λ, whose elements may take any real value, induces skewness in the return of individual assets. Adcock and Shutes (2012) describe multivariate versions of the normal-exponential and normal-gamma distributions. These are specific cases of the model proposed by Simaan (1987, 1993), which have not appeared explicitly in the literature before.

The paper presents extensions to Stein's lemma for both the MEST and CEST distributions. This work extends Adcock (2007), which presents versions of Stein's lemma for the MESN and CESN distributions. Similar to the results for these skew-normal distributions, a consequence of the single truncated variable employed in the MEST distribution is that the lemma offers results which are not complicated to compute for a given function $h(\cdot)$. As is shown below, the extension to Stein's lemma for the MEST distribution leads to a single mean–variance–skewness surface for expected utility maximisers. The lemma also leads to insights into the nature of skewness preference under this distribution. For the CEST distribution, the extension to the lemma leads to a single mean–variance–skewness hyper-surface. For general applications, this particular extension to the lemma is mainly of theoretical interest. This is because of the necessity to compute integrals of multivariate Student-t distributions and because parameter estimation will not be a trivial task. However once parameter estimates are available, it is shown in Section 7 that portfolio selection may be performed using quadratic programming. Notwithstanding its complexities, and as discussed in Section 5, a case may be made for a useful role for the CEST distribution in finance. Accordingly use of this distribution is a topic for future research. The paper then presents some analogous results for Simaan-type models. Given the more general nature of the models that Simaan proposes, these results are less elegant mathematically than those for the MEST and CEST distributions. Nonetheless, they show that, even under more general conditions, there is a single mean–variance–skewness surface for one factor models. When the models are extended to include more than one non-negative variable, the single mean–variance–skewness hyper-surface arises. Thus, this papers shows that there is a comprehensive set of multivariate probability distributions which incorporate asymmetry and which lead to straightforward extensions to Markowitz' efficient frontier. It is shown in Section 7 that portfolios on these surfaces may also be determined by quadratic programming.