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Robust multiobjective optimization & applications in portfolio optimization

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ABSTRACT

Motivated by Markowitz portfolio optimization problems under uncertainty in the problem data, we consider general convex parametric multiobjective optimization problems under data uncertainty. For the first time, this uncertainty is treated by a *robust multiobjective formulation* in the gist of Ben-Tal and Nemirovski. For this novel formulation, we investigate its relationship to the original multiobjective formulation as well as to its scalarizations. Further, we provide a characterization of the location of the *robust Pareto frontier* with respect to the corresponding original Pareto frontier and show that standard techniques from multiobjective optimization can be employed to characterize this robust efficient frontier. We illustrate our results based on a standard mean–variance problem.

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1. Motivation and overview

Prompted by the well-known strong data dependency of mean–variance optimization, we investigate how to treat multiobjective optimization problems with uncertain parameters. The framework of multiobjective optimization, in which several conflicting objectives have to be minimized simultaneously, can be seen as the natural setting for portfolio optimization problems, as such problems invariably have to deal with the conflicting notions of revenue and risk. However, and equally naturally, these problems also have to deal with uncertainty: problem data (like expected future return and covariances of random variables) are not known precisely, and only estimates are available. We are thus facing multiobjective optimization problems with uncertain parameters.

In this paper, we want to follow the idea of the *robust counterpart approach* where an entire set of possible parameter realizations – called *uncertainty set* – is used for the optimization, but no assumptions about the distribution of the unknown parameters is needed, in contrast to many other robustification approaches. In the context of portfolio optimization, several authors have considered instances varying from theoretical to practical settings, amongst others Goldfarb and Iyengar (2003), Tütüncü and Koenig (2004), Ceria and Stubbs (2006), Meucci (2005), Lutgens (2004) or Schöttle and Werner (2006). Most of these approaches have in common that the uncertainty sets are chosen based on statistical reasoning, but other approaches are followed as well. All these approaches are based on just one specific choice of the risk-aversion parameter, i.e. only robustification of a particular instance of risk-return trade-off is considered.

In contrast to these approaches, we propose a different and completely novel approach. Instead of following Markowitz (1952) and refering to some scalar portfolio optimization problem, we start with the multiobjective formulation of the mean–variance portfolio problem in the gist of Kuhn and Tucker (1951). Our main aim is then to robustify the complete efficient frontier to obtain a *robust efficient frontier*. In this context, our main contributions are as follows:

- We introduce for the first time a robust counterpart to a multiobjective programming problem in the style of Ben-Tal and Nemirovski (1998, 1999).
- We investigate the relationship between the robust Pareto frontier and the original Pareto frontier and show that the robust frontier lies between the original nominal efficient frontier and some corresponding easy-to-determine upper bound.
- We demonstrate that robust efficient frontiers can be found by standard methods of robust and multiobjective programming under commonly made assumptions on the uncertainty.
- We pay particular attention to the case of portfolio optimization and show that the resulting robust multiobjective counterpart of the mean–variance portfolio optimization problem can be treated in a numerically efficient manner.

We want to emphasize that, although motivated by portfolio optimization, our methodology is general enough to be applied to any convex parametric multiobjective optimization problem under data uncertainty. As such, it might be especially useful for optimization problems arising in engineering.

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The rest of the paper is organized as follows: Section 2 gives an overview of the necessary machinery from multiobjective optimization (Section 2.2), robust optimization (Section 2.3) and (robust) portfolio optimization (Section 2.4) that we need further on. Readers already familiar with these topics can easily skip this part of the presentation. Section 3 then contains the main results of this paper: we provide a proper notion of a robustified multiobjective problem and show that it leads, via scalarization, to families of robust single-objective problems that can also be derived in a different way. We also provide an alternative motivation for the robust counterpart and characterize the location of the robust efficient frontier with respect to the original frontier. Section 4 considers an illustrative numerical example from portfolio optimization that provides further insight into the proposed robust counterpart. Finally, we conclude in Section 5.

2. Introduction to multiobjective and robust optimization

2.1. Notation

The following notation is used throughout. We use $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$ and $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ for the set of non-negative resp. strictly positive real numbers. $\mathcal{K}(M)$ denotes the space of all non-empty, convex and compact subsets of a given non-empty and convex set $M \subset \mathbb{R}^n$. Further, $B_\varepsilon(x)$ denotes the closed ε-ball around some vector $x \in \mathbb{R}^n$, $||x||$ denotes the 2-norm for $x \in \mathbb{R}^n$ and $||\Sigma||$ denotes the Frobenius norm for $\Sigma \in \mathbb{S}^n$, where $\mathbb{S}^n$ is the space of all symmetric $n \times n$ matrices and $\mathbb{S}^n_+$ is the cone of positive semidefinite matrices.

2.2. Multiobjective optimization

In this section we provide a brief introduction of the concepts in multiobjective optimization that are used in this paper. We follow closely the exposition given by Fliege and Vicente (2006).

In multiobjective optimization, several functions

$$f_1, \ldots, f_p : \mathbb{R}^n \longrightarrow \mathbb{R}$$

with $p > 1$ have to be minimized simultaneously over a set of feasible points characterized by a (convex) compact set $X \in \mathcal{K}(\mathbb{R}^n)$. The general problem can be conveniently stated in the form

$$\begin{aligned} &\text{efmin} \quad f : \mathbb{R}^n \longrightarrow \mathbb{R}^p \\ &\text{s.t.} \qquad x \in X, \end{aligned} \tag{M}$$

where $f = (f_1, \ldots, f_p)^\top$, and the exact meaning of "efmin" still has to be specified. We will do so in what follows.

Remark 2.1. Multicriteria optimization is the ideal setting to analyse portfolio optimization problems in the sense of Markowitz. If we work in a financial market with n risky assets and $x \in \mathbb{R}^n$ is a portfolio vector $(\sum_{i=1}^n x_i = 1)$, we can simply set $p = 2$, let, say, $f_1(x) = s(x) = x^\top \Sigma x$ be the risk function for some covariance matrix $\Sigma \in \mathbb{S}^n_+$ and let $f_2(x) = -m(x) = -\mu^\top x$ be the return function for some vector of expected returns $\mu \in \mathbb{R}^n$.

The reason for the formulation given here is that there is no standard total order for the image space $\mathbb{R}^p$. In contrast to this situation, in the classical single-objective case one always uses the standard total order defined by $x < y :\leftrightarrow y - x \in \mathbb{R}_{++}$ $(x, y \in \mathbb{R})$. Nevertheless, the idea of specifying an order by using a specific set defining it can be conveniently employed in multiobjective optimization, as the following discussion will show.

If an arbitrary order relation $\prec$ on $\mathbb{R}^p$ and a set $A \subseteq \mathbb{R}^p$ are given, the vector $a \in \mathbb{R}^p$ is called *minimal* or a *minimizer* w.r.t. $\prec$ in A if $a \in A$ and $a \preceq b$ for all $b \in A$. (Here, $\preceq$ is the reflexive hull of $\prec$, i.e., $a \preceq b$ if and only if $a = b$ or $a \prec b$.) Minimal points usually do not exist, one reason being that it is seldom the case that $\preceq$ is a total order. A weaker concept, the concept of *domination* is therefore needed. A point a *dominates* a point b, if $a \prec b$ and $a \neq b$ holds. A point a is *nondominated* in A, if $a \in A$ and there does not exist a point $c \in A$ with $c \prec a$ and $c \neq a$. This approach raises the question about which of the many orders in $\mathbb{R}^p$ one should choose when solving multicriteria problems.

Let $K \subseteq \mathbb{R}^p$ be an arbitrary set. Define $A := f(X)$ and the order

$$x <_K y :\leftrightarrow y - x \in K. \tag{2.1}$$

For such an order relation, define further

$$\text{eff}_K(A) := \{a \in A | a \text{ nondominated in } A \text{ w.r.t} <_K\},$$

the set of all *nondominated* or *efficient points* of the set A. It is this notion of efficiency or optimality that we will use when we speak of solutions of (M), *and the operator* efmin *in* (M) *is understood to search for such efficient points: the set of solutions of the problem* (M) *is the preimage of all nondominated points of the set* $A = f(X)$ *with respect to the order* $<_K$.

The next theorem is well known, see, e.g. Göpfert and Nehse (1990) or Fliege and Vicente (2006).

Theorem 2.2. *Let* $K \subset \mathbb{R}^p$ *be a set and let* $<_K$ *be the binary relation defined by* K *as in* (2.1). *Then, the following statements hold:*

1. *If* $0 \in K$ *then* $<_K$ *is reflexive.*
2. *If* $K + K \subseteq K$ *then* $<_K$ *is transitive.*
3. *If* K *is a cone containing no lines, i.e.,* $K \cap -K \subseteq \{0\}$, *then* $<_K$ *is anti-symmetric. (In this case, the set* K *is also called pointed.)*
4. *The order* $<_K$ *is total if and only if* $K \cup -K = \mathbb{R}^p$.
5. *The set* K *is closed if and only if the relation* $<_K$ *is "continuous at 0" in the following sense. For all* $a \in \mathbb{R}^p$ *and all sequences* $(a^{(i)})_{i\in\mathbb{N}}$ *in* $\mathbb{R}^p$ *with* $\lim_{i\to+\infty} a^{(i)} = a$ *and* $0 <_K a^{(i)}$ *for all* $i \in \mathbb{N}$ *it follows that* $0 <_K a$ *holds.*

Note that $K + K \subseteq K$ holds if K is a convex cone. According to the theorem above, practicioners prefer to choose a closed convex cone K with $0 \in K$ which contains no lines to define the partial order $<_K$. (Note that the lexicographic order in $\mathbb{R}^p$ is defined by a cone which is not closed.) Moreover, in our context the space $\mathbb{R}^p$ will be the image space of functions to be minimized. As a consequence, it is important for numerical reasons to have scale-invariance of the induced order. This means that if $x <_K y$ and $\lambda > 0$ then $\lambda x <_K \lambda y$, a property which holds if and only if the set K is a cone.

Using a fixed set K to define an order relation as in (2.1) has one additional advantage. For an arbitrary relation $\prec$, the sets

$$C(a) := \{b \in \mathbb{R}^p | a \prec b\} - a \tag{2.2}$$

are constant if there exists a set K such that $\prec = <_K$ holds. Indeed, if $\prec = <_K$ then $C(a) = \{b \in \mathbb{R}^p | a \prec b\} = \{b \in \mathbb{R}^p | a <_K b\} = \{b \in \mathbb{R}^p | b - a \in K\} = a + K$. This means that $\prec$ is translation-invariant, i.e., $x + z \prec y + z$ for all z if and only if $x \prec y$.

To summarize the discussion above: *we need to choose a convex cone* $K \subset \mathbb{R}^p$ *with* $0 \in K$ *in order to define an order* $<_K$. *Other attributes of* K *that can be used to our advantage are closedness, pointedness, and* $K \cup -K = \mathbb{R}^p$ *but, as pointed out before, we cannot have all of these at the same time.*

Often, K is chosen to be the positive orthant, $K = \mathbb{R}^p_+$, which gives exactly the standard definition of order in multicriteria optimization and especially in the specific setting of mean–variance optimization.

We will now consider linear forms from $\text{int}(K^*)$, where K^* is the *dual cone* of K defined by $K^* = \{\lambda | \forall a \in K : \lambda^\top a \geqslant 0\}$. It turns out that in the case of convex cones and sets, these linear forms can

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