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Optimal multi-period mean–variance policy under no-shorting constraint

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ABSTRACT

We consider in this paper the mean–variance formulation in multi-period portfolio selection under no-shorting constraint. Recognizing the structure of a piecewise quadratic value function, we prove that the optimal portfolio policy is piecewise linear with respect to the current wealth level, and derive the semi-analytical expression of the piecewise quadratic value function. One prominent feature of our findings is the identification of a deterministic time-varying threshold for the wealth process and its implications for market settings. We also generalize our results in the mean–variance formulation to utility maximization with no-shorting constraint.

1. Introduction

In 1952, Markowitz (1952) pioneered the investment science and, in a broader sense, the modern finance, by considering the static mean–variance portfolio selection formulation in a market in which shorting is not allowed. Furthermore, he devised a numerical scheme, the critical line algorithm, to solve the static mean–variance model with no-shorting (see Markowitz (1952) and Markowitz (1956)). On the other hand, considering markets with shorting allowed leads to an unconstrained mean–variance portfolio selection formulation and facilitates the derivation of an analytical solution. More specifically, Merton (1972) applied unconstrained convex quadratic programming technique to obtain the analytical forms of the portfolio policy and the mean–variance efficient frontier for unconstrained mean–variance portfolio selection formulation. Twenty-eight years after the work by Merton (1972), Li and Ng (2000) and Zhou and Li (2000) successfully extended the unconstrained mean–variance portfolio selection formulation to the multi-period setting and to the continuous time setting, respectively. One prominent feature of the dynamic mean–variance formulations is that the optimal portfolio policy is always linear with respect to the current wealth.

After the paper by Li and Ng (2000) was published, Professor Markowitz wrote to one of the authors of this paper with a suggestion to extend the results in Li and Ng (2000) to the dynamic mean–variance formulation with a no-shorting constraint, and offered a conjecture of a piece-wise quadratic value function for such a situation (Markowitz, 2000). Stimulated by Prof. Markowitz's comments, Li et al. (2002) derived the optimal portfolio policy for the continuous-time mean–variance model with no-shorting using the duality method (Cvitanić and Karatzas, 1992; Xu and Shreve, 1992; Xu and Shreve, 1992), thus partially proving Prof. Markowitz's conjecture for the continuous-time mean–variance model. However, it has taken us more than 10 years in exactly proving Prof. Markowitz's conjecture for the discrete-time mean–variance formulation with no-shorting and in reporting our results in this paper. Only up to this stage, we finally recognize some inherent difference between the continuous-time and discrete-time formulations with a no-shorting constraint. In Li et al. (2002), the price processes of assets are continuous Itô processes. We will reveal in this paper that, due to the continuous adjustment of the portfolio policy, the wealth process in continuous time under the optimal policy without shorting behaves regularly below a deterministic time-varying threshold, thus retaining the same structure of one-piece quadratic value function as in the case with shorting allowed. On the contrary, the discontinuity of the wealth process in the discrete-time formulation leads to an actual situation with a piece-wise quadratic value function, as we will discuss in this paper. The existence of multiple pieces of quadratic value functions indeed poses a significant obstacle for us to achieve an analytical solution.

The past 10 years have also witnessed some other extensions of the mean–variance portfolio selection formulation with different constraints, see for examples, Bielecki et al. (2005) considered bankruptcy prohibition in continuous time using

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martingale approach; Zhu et al. (2004) investigated risk control of bankruptcy in discrete time; Sun and Wang (2006) studied a market consisting of a riskless asset and one risky portfolio under constraints such as market incompleteness, no-shorting, or partial information; Labbé and Heunis (2007) and Czichowsky and Schweizer (2010) suggested dual formulations to characterize the mean–variance portfolio selection and mean–variance hedging with general convex constraints respectively, without giving corresponding analytical solution; and Fu et al. (2010) integrated the borrowing rate constraints in the dynamic mean–variance model. Please also refer to Chiu and Wong (2012), Gulpinar and Rustem (2007), Li et al. (2010) and Wang and Forsyth (2011) for other interesting developments in the literature. Recently, Czichowsky and Schweizer (2011) further considered general cone-constrained continuous-time mean–variance portfolio selection with price processes being semimartingales. They also found that the value function is piecewise quadratic in such a case, which verifies that the feature of a piecewise quadratic value function roots from the discontinuity of price processes as we stated above in this paper. Different from Czichowsky and Schweizer (2011), we further find in this paper the fact that the discontinuity of the price processes may *not* always lead to a piecewise quadratic value function. Actually, as revealed in Section 3.3, under some types of bounded requirements on assets' return, the mean–variance formulation may still have a one-piece quadratic value function in discrete time. Such a finding gives us a much clearer view on the influence of market setting. Moreover, we also derive semi-analytical solutions for the expected utility maximization problems under no-shorting constraint.

The remaining of this paper is organized as follows. In Section 2, we present the mean–variance formulation for multi-period portfolio selection under no-shorting constraint. We derive in Section 3 the semi-analytical solution to the multi-period mean–variance formulation without shorting, study the properties of the solution, compare the results in this paper with the results in Li et al. (2002) for continuous-time mean–variance formulation with no-shorting, and investigate a particular market setting in which the piecewise quadratic value function reduces to the one-piece quadratic one. In Section 4, we extend our results to expected utility maximization with no-shorting. We demonstrate the solution scheme developed in this paper and investigate the inherent difference between the continuous time and discrete time via illustrative examples in Section 5. Finally, we conclude our paper in Section 6.

2. Mean–variance formulation for multi-period portfolio selection without shorting

The capital market under consideration consists of n risky assets with random rates of returns and one riskless asset with a deterministic rate of return. An investor with an initial wealth x_0 joins the market at time 0 and allocates his wealth among the $(n+1)$ assets. He can reallocate his wealth among the $(n+1)$ assets at the beginning of each of the following $(T-1)$ consecutive time periods. The deterministic rate of return of the riskless asset at time period t is denoted by $s_t > 0$ and the rates of return of the risky assets at time period t are denoted by a random vector $\mathbf{e}_t = [e_t^1, \dots, e_t^n]'$, where e_t^i is the random return for asset i at time period t. While assuming in this paper that vectors $\mathbf{e}_t$, $t = 0, 1, \dots, T-1$, are statistically independent, the only information we need to know for return $\mathbf{e}_t$ is its mean vector, $\mathbb{E}[\mathbf{e}_t] = [\mathbb{E}[e_t^1], \dots, \mathbb{E}[e_t^n]]'$ and its covariance, Cov $(\mathbf{e}_t)$, which is assumed to be positive definite.

Let x_t be the wealth of the investor at the beginning of the tth time period, and u_t^i, $i = 1, 2, \dots, n$, be the amount invested in the ith risky asset at the beginning of the tth time period. It is assumed that the short selling is not allowed, i.e., $\mathbf{u}_t = [u_t^1, u_t^2, \dots, u_t^n]' \in \mathbb{R}_+^n$ for $t = 0, 1, 2, \dots, T-1$. An investor of mean–variance type is seeking the best feasible investment strategy, $\{\mathbf{u}_t^\star\}_{t=0}^{T-1}$, such that the variance of the terminal wealth, Var (x_T), is minimized subject to that the expected terminal wealth, $\mathbb{E}[x_T]$, is set at a preselected level $d \geqslant x_0 \prod_{t=0}^{T-1} s_t$,

$$
\begin{aligned}
(P(d)): \quad & \min \quad \mathrm{Var}(x_T) \equiv \mathbb{E}[(x_T - d)^2], \\
& \text{s.t.} \quad \mathbb{E}[x_T] = d, \\
& \qquad x_{t+1} = s_t x_t + \mathbf{P}_t' \mathbf{u}_t, \\
& \qquad \mathbf{u}_t \geqslant \mathbf{0}, \quad t = 0, 1, \cdots, T-1,
\end{aligned}
\tag{1}
$$

where $\mathbf{P}_t = \left[P_t^1, P_t^2, \dots, P_t^n\right]' = \left[(e_t^1 - s_t), (e_t^2 - s_t), \dots, (e_t^n - s_t)\right]'$ is the vector of excess rates of returns and $\mathbf{0}$ denotes the n-dimensional zero vector. It is also reasonable to assume that $\mathbb{E}[\mathbf{P}_t]$ is a positive vector. The information set at the beginning of the t-th time period is denoted as

$$\mathcal{F}_t = \sigma(x_0, x_1, \cdots, x_t) = \sigma(\mathcal{F}_0 \vee \sigma(\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_{t-1}, \mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{t-1})),$$

where $\mathcal{F}_0$ contains x_0, s_t, $\mathbb{E}[\mathbf{e}_t]$ and Cov $(\mathbf{e}_t)$, $t = 1, \dots, T-1$. A feasible investment strategy at time period t, $\mathbf{u}_t$, is confined to be $\mathcal{F}_t$-measurable.

Due to Cov $(\mathbf{e}_t) \succ 0$, the second moment of $(s_t, \mathbf{e}_t')'$ is positive definite for all time periods. The following is then true for $t = 0, 1, \dots, T-1$:

$$
\begin{bmatrix} s_t^2 & s_t \mathbb{E}[\mathbf{P}_t'] \\ s_t \mathbb{E}[\mathbf{P}_t] & \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] \end{bmatrix}
=
\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & \cdots & 1 \end{bmatrix}
\begin{bmatrix} s_t^2 & s_t \mathbb{E}[\mathbf{e}_t'] \\ s_t \mathbb{E}[\mathbf{e}_t] & \mathbb{E}[\mathbf{e}_t \mathbf{e}_t'] \end{bmatrix}
\begin{bmatrix} 1 & -1 & \cdots & -1 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}
\succ 0,
$$

which implies $\mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] \succ 0$ and

$$s_t^2 (1 - \mathbb{E}[\mathbf{P}_t'] \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t]) > 0 \tag{2}$$

for all $t = 0, 1, \dots, T-1$. Varying parameter d in the problem formulation $(P(d))$ yields the efficient frontier in the mean–variance space.

3. Optimal multi-period mean–variance policy with no-shorting

3.1. Main result

Consider an auxiliary problem of $(P(d))$ by introducing Lagrangian multiplier 2μ,