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Inverse portfolio problem with mean-deviation model

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ABSTRACT

A Markowitz-type portfolio selection problem is to minimize a deviation measure of portfolio rate of return subject to constraints on portfolio budget and on desired expected return. In this context, the inverse portfolio problem is finding a deviation measure by observing the optimal mean-deviation portfolio that an investor holds. Necessary and sufficient conditions for the existence of such a deviation measure are established. It is shown that if the deviation measure exists, it can be chosen in the form of a mixed CVaR-deviation, and in the case of n risky assets available for investment (to form a portfolio), it is determined by a combination of $(n+1)$ CVaR-deviations. In the later case, an algorithm for constructing the deviation measure is presented, and if the number of CVaR-deviations is constrained, an approximate mixed CVaR-deviation is offered as well. The solution of the inverse portfolio problem may not be unique, and the investor can opt for the most conservative one, which has a simple closed-form representation.

1. Introduction

Understanding and modeling of individual risk preferences has long been a central venue in decision sciences, finance, and economics. A general approach is to introduce a system of axioms on preference relations that agree with "rational" behavior and then to construct a numerical equivalent (representation) to that system, so that given a choice of several random variables, a decision maker compares only their numerical equivalents. For example, the celebrated von Neumann and Morgenstern's theory [12] introduces axioms of completeness, transitivity, continuity, and independence and shows that the decision maker needs to deal only with expected utilities. A change in the system of axioms leads to another numerical equivalent that results in possibly different decisions. Since Neumann and Morgenstern's seminal work [12], similar choice theories such as the prospect theory [7], the dual utility theory [20], and the regret theory have emerged. However, none of their axiomatic systems are in complete agreement with empirical evidence, which is known as axiom violations or paradoxes. Even if a particular decision maker does agree on a proposed system of axioms on preference relations, still there is a wide class of possible numerical equivalents (utility functions) that represent the system, so that finding a particular numerical equivalent amounts to a tedious questionnaire procedure. A different approach was proposed by Markowitz [10], who suggested that portfolios of risky assets can be ordered at least partially based on two quantities: mean and standard deviation of portfolio rate of return. The approach reduces mean–variance portfolio selection to a simple quadratic programming problem and also leads to a convenient capital asset pricing model (CAPM). However, the simplicity of this approach has proved to be both its advantage and disadvantage. The extensive body of empirical and theoretical research shows that standard deviation is hardly an appropriate measure of risk and that a market portfolio, predicted by the mean–variance approach to be held by all investors, is only a theoretical concept (a real market index does not follow from the mean–variance approach). General measures of deviation introduced by Rockafellar et al. [14,15] address main shortcomings of standard deviation: they are not symmetrical with respect to ups and downs of a random variable and provide sufficient flexibility in customizing individual risk preferences. Let a random variable X represent the portfolio return. A Markowitz-type portfolio selection problem replaces standard deviation by a general deviation measure $\mathcal{D}$:

$$\min_{X\in\mathcal{V}} \mathcal{D}(X) \quad \text{subject to} \quad EX \geqslant \pi, \tag{1}$$

where π is a desired expected return, and $\mathcal{V}$ is a feasible set of portfolio returns. Rockafellar et al. [17,16] generalized the one-fund theorem and CAPM, originally established for (1) with standard deviation, for an arbitrary deviation measure. Also, if each investor believes in the mean-deviation paradigm [4] and has a corresponding deviation measure as a numerical representation of his/her risk preferences then there exists a market equilibrium (see [18]), and, moreover, a group of investors can form a cooperative portfolio that satisfies (1) with a "cooperative" deviation measure (see [5]).

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However, as in the aforementioned utility theories and in contrast to the Markowitz's mean–variance approach, a variety of deviation measures brings the question of choosing appropriate deviation measure for each investor. Partially, this question was answered in [3] by establishing a one-to-one correspondence between the class of alpha-concave distributions and the class of comonotone deviation measures via the maximum entropy principle. However, the resulting deviation measure reflected the "average" preferences (of the market as a whole) rather than preferences of an individual investor.

A different perspective to this question could be offered through an inverse approach. Instead of agonizing over idealized postulates of rational behavior, an investor can identify his/her risk preferences in the form of a set of benchmarks (portfolios, financial instruments, assets, etc.) that he/she is relatively satisfied with. This work solves the inverse portfolio problem: given a convex set of bounded portfolio returns $\mathcal{V}$ and a portfolio return $X^* \in \mathcal{V}$, find a deviation measure $\mathcal{D}^*$ such that X^* is optimal for (1) with $\mathcal{D}^*$. An approximate solution to this problem was given in [8], where a set of feasible deviation measures was confined to linear combinations of conditional value-at-risk (CVaR) deviations for certain risk-tolerance levels or, equivalently, to so-called *mixed CVaR-deviations*. However, an approximate $\mathcal{D}^*$ substantially depends on the choice of the risk-tolerance levels and on an error measure for residuals. Here, it is proved that "exact" $\mathcal{D}^*$ exists for every risk-averse investor,[1] and that, indeed, the set for $\mathcal{D}^*$ can be narrowed down to mixed CVaR-deviations. However, $\mathcal{D}^*$ may not be unique, and it is suggested that in this case, the investor takes the supremum over all possible deviation measures $\mathcal{D}$, such that X^* is optimal for (1) with $\mathcal{D}$. Remarkably, this supremum admits a simple closed-form representation in the form similar to a *worst-case mixed-CVaR deviation*. In addition, given T scenarios for asset returns, an algorithm for finding $\mathcal{D}^*$ in (1) in the form of mixed CVaR-deviations is provided, and if the number of CVaR-deviations is constrained, an alternative approach to constructing an approximate deviation measure is also presented.

This work is organized into seven sections. Section 2 reviews basic notions of general deviation measures. Section 3 formulates and solves the inverse portfolio problem. Section 4 addresses the inverse problem for scenario-based portfolio selection. Section 5 finds an approximate deviation measure in the form of mixed CVaR-deviation with specified number of risk-tolerance levels. Section 6 constructs a risk-envelope representation for a non law-invariant deviation measure provided that law-invariant one does not exists. Section 7 concludes the work.

2. Deviation measures

Let $(\Omega, \mathcal{M}, \mathbb{P})$ be a probability space, where Ω denotes the designated space of future states ω, $\mathcal{M}$ is a field of sets in Ω, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{M})$. A random variable (r.v.) is considered to be an element of $\mathcal{L}^2(\Omega) = \mathcal{L}^2(\Omega, \mathcal{M}, \mathbb{P})$. $F_X(x)$ and $q_X(\alpha) = \inf\{x | F_X(x) > \alpha\}$ will denote the cumulative distribution function (CDF) and quantile function of an r.v. X, respectively. Also, C will denote a constant in the real numbers. The relations between r.v.'s are understood to hold in the almost sure sense, e.g., we write $X = Y$ if $\mathbb{P}[X = Y] = 1$ and $X \geqslant Y$ if $\mathbb{P}[X \geqslant Y] = 1$. The probability space Ω is called *atomless*, if there exists an r.v. with a continuous CDF. This implies existence of r.v.'s on Ω with all possible CDFs (see e.g. [2]).

[1] An investor is risk-averse, if the return X^* of the portfolio he/she holds is undominated in the sense of second-order stochastic dominance (SSD); see Section 2 for the exact definition.

Definition 1 (*general deviation measures*). A *deviation measure* is any functional $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0, \infty]$ satisfying[2]

(D1) $\mathcal{D}(X) = 0$ for constant X, but $\mathcal{D}(X) > 0$ otherwise (*nonnegativity*),
(D2) $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$ (*positive homogeneity*),
(D3) $\mathcal{D}(X + Y) \leqslant \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y (*subadditivity*),
(D4) set $\{X \in \mathcal{L}^2(\Omega) | \mathcal{D}(X) \leqslant C\}$ is closed for all $C < \infty$ (*lower semicontinuity*).

As shown in [14], axioms D1–D3 imply that

$$\mathcal{D}(X + C) = \mathcal{D}(X) \text{ for all constants } C \text{ (constant translation invariance)}. \quad (2)$$

In general, for two r.v.'s with the same CDF, a deviation measure may assume different values. This work is confined to law-invariant deviation measures [15], i.e., those which depend only on the CDF of an r.v.

Definition 2 (*law-invariant deviation measures*). A deviation measure $\mathcal{D}(X)$ is called *law-invariant*, if $\mathcal{D}(X_1) = \mathcal{D}(X_2)$ for any two r.v.'s X_1 and X_2 yielding the same CDF on $(-\infty; \infty)$.

Well-known examples of deviation measures include (see [15,14]):

(i) deviation measures of $\mathcal{L}^p$ type $\mathcal{D}(X) = \|X - EX\|_p, p \in [1, \infty]$, for example, the standard deviation $\sigma(X) = \|X - EX\|_2$ and mean absolute deviation $\mathrm{MAD}(X) = \|X - EX\|_1$, where $\|\cdot\|_p$ is the $\mathcal{L}^p$ norm;
(ii) deviation measures of semi-$\mathcal{L}^p$ type $\mathcal{D}_-(X) = \|[X - EX]_-\|_p$ and $\mathcal{D}_+(X) = \|[X - EX]_+\|_p$, $p \in [1, \infty]$, for example, *standard lower semideviation* $\sigma_-(X) = \|[X - EX]_-\|_2$ and *standard upper semideviation* $\sigma_+(X) = \|[X - EX]_+\|_2$, where $[X]_\pm = \max\{0, \pm X\}$;
(iii) range-based deviations, for example, lower range of X, defined as $\mathcal{D}(X) = EX - \inf X$, and its reflection, upper range of X, defined as $\mathcal{D}(X) = \sup X - EX$.
(iv) conditional value-at-risk (CVaR) deviation, defined for any $\alpha \in (0, 1)$ by

$$\mathrm{CVaR}_\alpha^\Delta(X) \equiv EX - \frac{1}{\alpha} \int_0^\alpha q_X(\beta)\, d\beta. \quad (3)$$

For convenience, let $\mathrm{CVaR}_0^\Delta(X) = EX - \inf X$ and $\mathrm{CVaR}_1^\Delta(X) = \sup X - EX$.[3]

As generalizations of CVaR deviation, Rockafellar et al. [15,14] introduced *mixed CVaR-deviation*

$$\mathcal{D}(X) = \int_0^1 \mathrm{CVaR}_\alpha^\Delta(X)\, d\lambda(\alpha) \quad (4)$$

with some $\lambda(\alpha) \geqslant 0$ such that $\int_0^1 d\lambda(\alpha) = 1$, and *worst-case mixed-CVaR deviation*

$$\mathcal{D}(X) = \sup_{\lambda \in \Lambda} \int_0^1 \mathrm{CVaR}_\alpha^\Delta(X)\, d\lambda(\alpha) \quad (5)$$

for some collection Λ of weighting nonnegative measures λ on $[0,1]$[4] with $\int_0^1 d\lambda(\alpha) = 1$. All the above deviation measures are law invariant. See [15,16] for other examples.

[2] In [14,15], axiom D4 was not included in the definition. Deviation measures satisfying D4 were called *lower semicontinuous* deviation measures.
[3] Observe that $\lim_{\alpha \to 1-} \mathrm{CVaR}_\alpha^\Delta(X) = 0$, whereas $\lim_{\alpha \to 1-} \frac{d}{d\alpha} \mathrm{CVaR}_\alpha^\Delta(X) = EX - \sup X$, so that $\mathrm{CVaR}_1^\Delta(X) = \sup X - EX$ is only a mathematical convention.
[4] In [15,14], nonnegative measures λ are defined on $(0,1)$, so that range-based deviations are not included into mixed CVaR-deviations.