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Worst-case robust Omega ratio

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ABSTRACT

The Omega ratio is a recent performance measure proposed to overcome the known shortcomings of the Sharpe ratio. Until recently, the Omega ratio was thought to be computationally intractable, and research was focused on heuristic optimization procedures. We have shown elsewhere that the Omega ratio optimization is equivalent to a linear program and hence can be solved exactly in polynomial time. This permits the investigation of more complex and realistic variants of the problem. The standard formulation of the Omega ratio requires perfect information for the probability distribution of the asset returns. In this paper, we investigate the problem arising from the probability distribution of the asset returns being only partially known. We introduce the robust variant of the conventional Omega ratio that hedges against uncertainty in the probability distribution. We examine the worst-case Omega ratio optimization problem under three types of uncertainty – mixture distribution, box and ellipsoidal uncertainty – and show that the problem remains tractable.

1. Introduction

The seminal approach of Markowitz (1952) changed and motivated the direction of research on portfolio selection. It is known that his proposed framework is based on assumptions that do not hold in practice. For instance, the Markowitz framework considers only the first two moments of the distribution and therefore implies that the underlying asset returns are normally distributed. However, the novel concept that the wealth allocation should be based on the statistical properties of assets and the risk return trade-off are now regarded fundamental. On this principle, a number of frameworks emerged that required less restrictive and more realistic assumptions.

Consider a single-period investor who wants to maximize his wealth at the end of the horizon, for a given level of risk; by allocating the initial wealth to the available securities. A natural choice for the investor is to formulate a utility maximization problem. This entails assumptions on the utility function of the investor and generally lies outside the scope of this paper. The second choice is to minimize the risk for a required return, or equivalently to maximize the expected return for a given level of risk. Thus, there is an inherent trade-off between risk and return. Whether the investor should accept a marginal increase of the risk to achieve an increase in the expected return has led to the consideration of performance measures. The Sharpe ratio is the first attempt to quantify the trade-off between risk and reward in investment under uncertainty. However, its underlying assumptions have been widely criticized (Lo, 2002). Alternative performance measures relax the strict assumptions of the Sharpe ratio (Sortino and Lee (1994), Zakamouline and Koekebakker (2009), Keating and Shadwick (2002)). In this paper we are concerned with one of these, namely the Omega ratio (Keating and Shadwick, 2002).

The Omega ratio entails the partitioning of the returns into losses and gains in excess of a predetermined threshold and it is defined as the probability weighted gains by the probability weighted losses. For a formal definition please refer to Eq. (1). The Omega ratio can be simplified to the ratio of the expected return in excess of the threshold by the first order lower partial moment. In other words, the Omega ratio considers the first order lower partial moment as a risk measure.

Lower partial moments have been already used as risk measures (Nawrocki (1999), Ogryczak and Ruszczynski (1999)). Unlike the variance, the first order lower partial moment can cope with skewed returns and therefore more suitable for real life applications. Empirical studies suggest that return distributions deviate significantly from the normal distribution, especially for commodities and alternative investments (Deaton and Laroque (1992), Brooks and Kat (2002)). In addition, the first order lower partial moment is linked with the second order stochastic dominance, which is based on the axiom of risk aversion (Ogryczak and Ruszczynski (2002a), Ogryczak and Ruszczynski (2002b)).

Kapsos et al. (2012) show that the optimal Omega ratio can be computed in polynomial time, using the fact that the Omega ratio is a quasiconcave function. This allows the study of more complex related issues. In theory, the Omega ratio can be used for any distribution of asset returns. Nevertheless, it assumes precise

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knowledge of the distribution. This is usually inferred from historical data or is specified by an expert. Unfortunately, the optimization of the Omega ratio can be severely biased and become over-optimistic when the probability distribution is only partially known, or corrupted by estimation errors.

In the presence of uncertain input parameters, the Omega ratio maximization problem has many solutions; one solution for each possible realization of the uncertain input. In order to address the above, we employ robust optimization and adopt a worst-case approach based on the assumption that only partial information is known regarding the uncertain inputs (Ben-Tal and Nemirovski (1998), Tütüncü and Koenig (2004)). In other words, instead of using a single value for the input parameters, we assume that the realization of the input parameters will be within an uncertainty set. This worst-case approach of the framework provides an immunization against the worst-case scenario for all possible realizations of the uncertain input. More specifically, we establish the worst-case Omega ratio maximization under a mixture distribution with uncertain mixing probabilities, box and ellipsoidal uncertainties. These type of uncertainty specifications have been examined under CVaR optimization by (Zhu and Fukushima (2009)). We show that the problem remains tractable and can be solved in polynomial time.

The rest of the paper is organized as follows. In Section 2, we show two methodologies that can be used to solve the standard Omega ratio maximization. In Section 3, we define the worst-case Omega ratio and show that its computation remains tractable under a mixture distribution with unknown mixing probabilities, ellipsoid and polyhedral uncertainty sets. Finally, in Section 4, we apply the worst-case Omega ratio maximization on artificial and market data.

2. Omega ratio optimization

Consider an investor that faces the wealth allocation problem with n securities. We denote the current time as $t = 0$ and $t = T$ as the end of the investment horizon. The portfolio is characterized by a vector of weights $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^n$, whose components add up to 1. The element x_i denotes the percentage of total wealth invested in the i^{th} asset at time $t = 0$. Let y_i denote the random return of asset i and the ith element of the vector $\boldsymbol{y} \in \mathbb{R}^n$. The random return of a portfolio of assets is given by $\boldsymbol{x}^\top \boldsymbol{y}$. With $f(y_i)$ and $F(y_i)$ we denote the probability density and cumulative distribution functions, respectively.

Keating and Shadwick (2002) define the Omega ratio as:

$$\Omega(y_i) = \frac{\int_\tau^{+\infty}[1 - F(y_i)]\,\mathrm{d}y_i}{\int_{-\infty}^{\tau} F(y_i)\,\mathrm{d}y_i}, \tag{1}$$

which can be simplified to

$$\Omega(y_i) = \frac{E(y_i) - \tau}{E[\tau - y_i]^+} + 1, \tag{2}$$

where τ a threshold that partitions the returns to desirable (gain) and undesirable (loss). The choice of the value for τ is left to the investor and can be set for example to be equal to the risk-free rate or zero. The Omega ratio is defined as the ratio of the area on the right of the threshold and above the cumulative distribution, over the area on the left of the threshold and below the cumulative distribution. We refer to Fig. 1 to illustrate the intuition behind the Omega ratio.

The Omega ratio makes use of the probability distribution of the underlying assets. Probability distributions can be classified into two major categories; continuous and discrete. A continuous probability distribution has no mass points, i.e. $Pr(y = u) = 0$. In contrast, a discrete probability distribution is characterized by a mass function, i.e. $\sum_u Pr(y = u) = 1$.

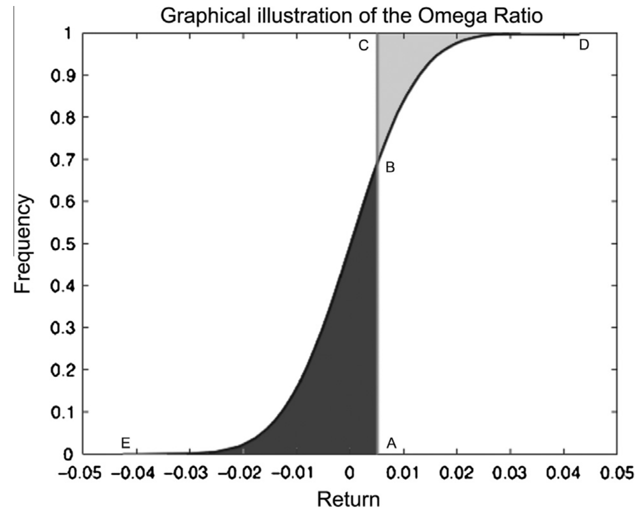

Fig. 1. Graphical illustration of the Omega ratio. The Omega ratio is defined as the ratio of the light gray area divided by the dark gray area.

Under a continuous probability distribution the Omega ratio for a portfolio is defined as

$$\Omega(x) = \frac{\boldsymbol{x}^\top E_p(\boldsymbol{y}) - \tau}{E_p([\tau - \boldsymbol{x}^\top \boldsymbol{y}]^+)} + 1, \tag{3}$$

whereas its discrete analog is

$$\Omega(x) = \frac{\boldsymbol{x}^\top (\boldsymbol{Y}^\top \boldsymbol{\pi}) - \tau}{\boldsymbol{\pi}^\top [\tau \mathbf{1} - (\boldsymbol{Y}\boldsymbol{x})]^+} + 1, \tag{4}$$

where $\boldsymbol{Y} \in \mathbb{R}^{S\times n}$ the matrix that contains the S sample returns for the n assets and $\boldsymbol{\pi}$ the vector with the probabilities for each sample return.

A reasonable criterion for an investor would be to hold the portfolio corresponding to the maximum Omega ratio. The Omega ratio has been used by a number of authors for portfolio construction in order to avoid the known limitations of the Sharpe ratio (Gilli et al. (2011), Kane et al. (2009), Passow (2005), Mausser et al. (2006)). Kapsos et al. (2012) have proposed two alternatives for the Omega maximization problem. For the case of the continuous probability distribution, they propose the use of an Efficient Frontier approach, whereas for the discrete case they employ linear-fractional programming.

2.1. Continuous distribution

According to Kapsos et al. (2012), the Omega maximization problem can be solved using the efficient frontier approach, i.e. by solving the following problem for different δ, evaluate the solutions and keep the solution with the maximum Omega ratio

$$\max_x \ \delta(\boldsymbol{x}^\top E_p(\boldsymbol{y}) - \tau) - (1-\delta)E_p([\tau - \boldsymbol{x}^\top \boldsymbol{y}]^+), \tag{5}$$

$$\text{s.t.} \ \boldsymbol{x} \in \mathcal{X}, \tag{6}$$

where $\mathcal{X}$ is a convex set that typically includes the budget constraint $\mathbf{1}^\top \boldsymbol{x} = 1$, upper and lower bounds on the weights, etc. The solution of the above sequence of problems yield the efficient Omega frontier. The portfolio with the optimal Omega ratio is given by the solution that corresponds to the maximum Omega ratio.

2.2. Discrete distribution

When the underlying distribution is discrete, the Omega maximization problem can be written as follows