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# Capital Asset Pricing Model (CAPM) with drawdown measure

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ABSTRACT

The notion of drawdown is central to active portfolio management. Conditional Drawdown-at-Risk (CDaR) is defined as the average of a specified percentage of the largest drawdowns over an investment horizon and includes maximum and average drawdowns as particular cases. The necessary optimality conditions for a portfolio optimization problem with CDaR yield the capital asset pricing model (CAPM) stated in both single and multiple sample-path settings. The drawdown beta in the CAPM has a simple interpretation and is evaluated for hedge fund indices from the HFRX database in the single sample-path setting. Drawdown alpha is introduced similarly to the alpha in the classical CAPM and is evaluated for the same hedge fund indices. Both drawdown beta and drawdown alpha are used to prioritize hedge fund strategies and to identify instruments for hedging against market drawdowns.



## 1. Introduction

The capital asset pricing model (CAPM) is one of the fundamental and most influential concepts in modern finance. It is closely related to portfolio theory and finds its application in portfolio risk management, fund performance measurement, security valuation, etc. The CAPM was developed by Sharpe [35], Lintner [16], and Mossin [22] and can be viewed from two perspectives:

(i) The CAPM is a reformulation of the necessary optimality conditions for the Markowitz's mean–variance portfolio problem and, thus, inherently depends on the definition of risk as variance.
(ii) The CAPM is a single-factor linear model (security market line) that relates the expected returns of an asset and a market portfolio, in which the slope, called asset *beta*, serves as a measure of asset non-diversifiable (systematic) risk.

Owing to its idealized assumptions, e.g. the use of variance as a measure of risk and homogeneity of risk preferences, the CAPM fails to "predict" stock returns accurately enough. Moreover, extensive empirical evidence shows that factors other than market portfolio contribute to stock return variations. Nevertheless, the CAPM, being mathematically simple, still offers a quick quantitative insight into risk-reward interplay and remains a major benchmark for asset pricing. Since its conception, the CAPM has been extended largely in three main directions: (i) relaxing or changing the assumptions under which the CAPM was derived, (ii) identifying new factors as explanatory variables for stock returns, and (iii) using different risk measures for portfolio valuation; see [23,12] for detailed CAPM reviews. Prominent CAPM extensions in the first direction include an intertemporal CAPM [20] and Black-Litterman model [3] that incorporates investor views on asset returns. Also, to lessen beta intertemporal instability, Levy [15] and Fabozzi and Francis [10] introduced several definitions of bear/bull market and applied the CAPM to different time periods separately. The second direction is aligned with the arbitrage pricing theory (APT), which explains asset returns through linear models with several factors and is often viewed as an empirical counterpart of the CAPM. Exemplary contributions to the APT are the works of Ross [32], Banz [2], Rosenberg [31], Burmeister and Wall [4], etc. to name just a few, whereas the Fama and French three-factor model [11] is now widely accepted as a CAPM empirical successor. Recently, Ranaldo [24] suggested higher distribution moments as factors for hedge fund pricing. As for the use of different risk measures in place of variance, Markowitz [18] himself acknowledged the shortcomings of variance and proposed the mean-semivariance approach to portfolio selection. In fact, standard deviation and lower semideviation are particular examples of general deviation measures [27] that are not necessarily symmetric with respect to ups and downs of a random variable and can be customized to tailor investor's risk preferences. Rockafellar et al. [28–30] developed mean-deviation approach to portfolio selection as an extension of the Markowitz's mean–variance ap-

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proach and generalized a number of classical results, including the one-fund theorem [28], the CAPM [29] and the existence of market equilibrium with investors using different deviation measures [30].

After 60 years of intensive CAPM research, the search for an informative and yet simple one-factor model explaining stock returns continues. So far, the theoretical approach through the portfolio theory to finding risk or deviation measures for a portfolio problem that would improve the CAPM has had limited success. This could be partially explained by the fact that one-period models ignore sequential structure of returns. For example, if pairs of returns for an asset and a market portfolio corresponding to different time moments are reshuffled, the covariance of the returns will not be affected. However, from the investment perspective, the original return sequence and the reshuffled one are unlikely to be viewed equally. In fact, investors are concerned not only about magnitude of asset losses but also about their sequence. One of the measures that capture this phenomenon is based on asset *drawdown*, which at each time moment is the drop of the asset cumulative return from its peak value since the time of investment or monitoring. Usually, investors quit an investment fund after either a single large drawdown (greater than 20%) or a small but prolonged drawdown (over a year). As a result, active portfolio management imposes several constraints on fund drawdowns [5]. Funds can be compared by Managed Account Reports (MAR) ratio, which is similar to Sharpe ratio [36] and is defined as the ratio of fund compound annual return from inception to the fund maximum drawdown from inception. However, the MAR ratio and similar Calmar (Sterling) ratio (also based on the notion of drawdown) do not involve market portfolio, and thus, are unable to identify funds producing positive returns when the market portfolio is in drawdown. In other words, these ratios are not substitutes for *drawdown beta*, which can identify instruments for hedging against market drawdowns.

The goal of this work is twofold: (i) to define drawdown beta $\beta_{DD}$ and to demonstrate empirically that $\beta_{DD}$ is an informative measure of asset performance for hedging against market drawdowns and (ii) to lay down a theoretical foundation for $\beta_{DD}$ through portfolio theory. Grossman and Zhou [13] were arguably the first to solve a portfolio problem with a constraint on portfolio *relative* drawdown with a single risky asset in a continuous-time setting, whereas Cvitanic and Karatzas [8] extended Grossman and Zhou's approach to the case of several risky assets, and recently, Yang and Zhong [37] adopted the Cvitanic and Karatzas' continuous-time solution to a discrete-time setting with a rolling time window. The line of research on portfolio optimization under drawdown constraints was further advanced in [34,7], where in particular, Cherny and Obloj [7] showed that in the setup of Grossman and Zhou, the problem of maximizing the long-term growth rate of the expected utility of wealth under a drawdown constraint could be reduced to an unconstrained problem with a modified utility function. Also, Elie and Touzi [9] studied an optimal consumption-investment problem with an infinite horizon and a drawdown constraint. Other relevant works on drawdown include capital allocation strategies with a drawdown constraint [21], mean–variance portfolio problem with drawdown constraints [1], drawdown modeling and estimation [19,17], and empirical studies [14]. At this point, it should be noted that for a given investment horizon, portfolio drawdowns are either a function in a continuous-time setting or a time series in a discrete-time setting, and while portfolio drawdowns can be constrained at each time period, they cannot be easily compared to drawdowns of other portfolios even for the same investment horizon. In other words, to characterize portfolio risk for the whole investment period, drawdowns (function or time series) should be translated into a single number, or, equivalently, into *drawdown measure*. For a single sample-path of portfolio return, Chekhlov et al. [5] proposed a drawdown measure, called *Conditional Drawdown-at-Risk (CDaR)*, as the average of a specified percentage of the largest drawdowns over the investment horizon. In the followup work, Chekhlov et al. [6] extended CDaR definition for the multiple sample-path case and formulated a portfolio optimization problem with CDaR, which was reduced to linear programming with the Rockafellar–Uryasev formula for *Conditional Value-at-Risk (CVaR)* [25,26]. In fact, CDaR possesses all the properties of a deviation measure: it is convex, positive homogeneous, nonnegative, and invariant to constant translation. This suggests that the approach used to derive the CAPM with general deviation measures in [28,29] can be applied to obtain the CAPM with CDaR. Several case studies on portfolio optimization with CDaR and real data from hedge funds are available at the University of Florida Financial Engineering Test Problems webpage.[1] They provide codes, data, and optimization results.

This work derives necessary optimality conditions for a portfolio optimization problem with CDaR, similar to the one addressed in [6], and reformulates the conditions in the form of CAPM yielding definition for the drawdown beta $\beta_{DD}$. The drawdown beta has a simple interpretation and depends on the confidence level $\alpha \in [0,1]$ that determines the percentage of the largest drawdowns in CDaR with $\alpha = 0$ and $\alpha = 1$ corresponding to the average and maximum drawdowns, respectively. The drawdown beta and classical beta have a clear distinction: drawdown beta accounts only for periods when the market portfolio is in drawdown and ignores asset performance when the market portfolio goes up, whereas the classical beta evaluates correlation between returns of the asset and market portfolio over the whole period. To some extent, $\beta_{DD}$ is similar to the bear market beta introduced in [10], which also concentrates on poor market performance, but in contrast to $\beta_{DD}$, has no portfolio optimization rationale.

The paper is organized into four sections. Section 2 formulates the portfolio optimization problem with CDaR and derives the necessary optimality conditions, which are then restated as CAPM in the multiple sample-path setting. Section 3 presents the CAPM with CDaR in the case of a single sample-path and evaluates $\beta_{DD}$ for hedge fund indices from the HFRX database. It also introduces the *drawdown alpha* (similar to the classical alpha) as asset excess return compared to the CAPM prediction and evaluates drawdown alphas for the same hedge fund indices. Section 4 concludes the paper.

## 2. Drawdown CAPM with CDaR: theoretical background

The classical CAPM establishes a linear relationship between the expected rate of return of an asset and the expected rate of return of a market portfolio, where the slope of the linear relationship is called asset beta and is defined as the ratio of the covariance of the asset and market portfolio rates to the variance of the market rate. If the asset beta and the expected rate of the market portfolio are known or given, the CAPM "predicts" the asset expected rate of return. However, asset beta can be defined differently thereby leading to a non-classical CAPM. In fact, it is closely related to a risk or deviation measure used in a portfolio selection problem yielding an optimal portfolio. This section formulates portfolio optimization problems with CDaR and presents their necessary optimality conditions in the form of the CAPM in the multiple sample-path setting.

[1] Case studies website: http://www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/, case study Portfolio Optimization with Drawdown Constraints on a Single Path, case study Portfolio Optimization with Drawdown Constraints on Multiple Paths, case study Portfolio Optimization with Drawdown Constraints, Single Path vs Multiple Paths.