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Twenty years of linear programming based portfolio optimization

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ABSTRACT

Markowitz formulated the portfolio optimization problem through two criteria: the expected return and the risk, as a measure of the variability of the return. The classical Markowitz model uses the variance as the risk measure and is a quadratic programming problem. Many attempts have been made to linearize the portfolio optimization problem. Several different risk measures have been proposed which are computationally attractive as (for discrete random variables) they give rise to linear programming (LP) problems. About twenty years ago, the mean absolute deviation (MAD) model drew a lot of attention resulting in much research and speeding up development of other LP models. Further, the LP models based on the conditional value at risk (CVaR) have a great impact on new developments in portfolio optimization during the first decade of the 21st century. The LP solvability may become relevant for real-life decisions when portfolios have to meet side constraints and take into account transaction costs or when large size instances have to be solved. In this paper we review the variety of LP solvable portfolio optimization models presented in the literature, the real features that have been modeled and the solution approaches to the resulting models, in most of the cases mixed integer linear programming (MILP) models. We also discuss the impact of the inclusion of the real features.

1. Introduction

The portfolio optimization problem considered in this paper follows the original Markowitz' formulation and is based on a single period model of investment. At the beginning of a period, an investor allocates the capital among various securities, assigning a share of the capital to each. During the investment period, the portfolio generates a random rate of return. This results in a new value of the capital (observed at the end of the period), increased or decreased with respect to the invested capital by the average portfolio return. This model has played a crucial role in stock investment and has served as basis for the development of the modern portfolio financial theory.

In the original Markowitz model (Markowitz, 1952) the risk is measured by the standard deviation or variance. Several other risk measures have been later considered, creating a family of mean-risk models. Whereas the original Markowitz model is a quadratic programming problem, following Sharpe (1971a), many attempts have been made to linearize the portfolio optimization problem (c.f., Speranza (1993) and references therein).

Nowadays, solution methods available for quadratic programming models are quite competitive also with respect to linear models. Nevertheless, the introduction of real features involving the use of integer variables may increase problem complexity significantly and makes LP solvable models more competitive with respect to quadratic models for which satisfactory solution methods are not available. Moreover, the recent advance in computers capability has opened up new solution opportunities and led to an extraordinary progress in statistics (see Efron (2000)) as well as in optimization (see Mulvey (2004) and Cornuejols & Tütüncü (2007)) with enormous effects in different application contexts including finance.

Obviously, in order to guarantee that the portfolio takes advantage of diversification, no risk measure can be a linear function of the portfolio shares. Nevertheless, a risk measure can be LP computable in the case of discrete random variables, when returns are defined by their realizations under the specified scenarios. This applies, in particular, to the mean absolute deviation from the mean. The mean absolute deviation was very early considered in the portfolio analysis (Sharpe (1971b) and references therein) while Konno and Yamazaki (1991) presented and analyzed the complete portfolio optimization model based on this risk measure – the so-called MAD model. The MAD model presented in 1991 was not the first LP portfolio optimization model as earlier Yitzhaki (1982) introduced the mean-risk model using Gini's mean

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(absolute) difference as risk measure. Nevertheless, the MAD model as much simpler computationally has drawn a lot of attention resulting in much research and speeding up development of other LP models. Young (1998) analyzed the LP solvable portfolio optimization model based on risk defined by the worst case scenario (minmax approach), while Ogryczak (2000) introduced the multiple criteria LP model covering all the above as special aggregation techniques. Following Rockafellar and Uryasev (2000, 2002), the CVaR models had a great impact on new developments of risk measures in finance during the first decade of 21st century. While several LP computable measures are dispersion type risk measures, some are safety measures which, when embedded in an optimization model, are maximized instead of being minimized. A first survey on risk and safety *basic* LP solvable portfolio optimization models can be found in Mansini, Ogryczak, and Speranza (2003a).

In practical financial applications the portfolio optimization problem has to take into account real features such as transaction costs, minimum transaction lots, cardinality constraints, thresholds on maximum or minimum investments. The impact of the introduction of real features in a portfolio optimization model on the resulting portfolio has been discussed in Kellerer, Mansini, and Speranza (2000), where it is shown on real data that the introduction of fixed transaction costs reduces the number of securities selected, and that considering transaction lots substantially changes the structure of the resulting portfolio, both in terms of securities selected and capital invested in the securities.

In most cases the inclusion of real features in a basic model requires the introduction of integer and binary variables. We refer to these models as models *with real features*. In some cases the modeling of real features is possible by using as decision variables the security shares (percentages). We call the models based on shares *relative models* and the investment variables *relative*. In several cases the introduction of real features implies the need of variables that represent the absolute values of the capital invested in each security. We call this second type of models *absolute models* and the investment variables *absolute*.

In this paper we review the basic LP solvable portfolio models and the models with real features that were presented in the literature, together with the solution approaches proposed for the latter class of models. Though optimization models are a consolidated approach to solve complex real problems, the relevance of heuristics has been also well recognized since the eighties (see Zanakis & Evans (1981)). The use of a heuristic, a threshold-accepting algorithm, for portfolio optimization is discussed in Dueck and Winker (1992). Since then, and in particular in the last decade thanks to the enormous growth in computing power, practitioners and financial firms have made a massive recourse to efficient and easy to implement heuristic techniques when performing strategic what-if analysis studies (see Gilli & Schumann (2012)). Besides, a new relevance in practical applications is obtained by approaches taking into account the multiple criteria nature of the portfolio problem (see the recent work by Xidonas, Mavrotas, Zopounidis, & Psarras (2011) for an integrated methodological framework for portfolio optimization based on multiple criteria decision making (MCDM), Zopounidis & Doumpos (2002) and Steuer & Na (2003), for literature reviews on multicriteria decision in financial decision making).

The paper is organized as follows. Section 2 is devoted to an introduction to risk and safety measures and reviews the basic LP solvable portfolio optimization models. In Section 3 we recall short fall risk measures as the basic LP computable risk measures. We also analyze mixed criteria obtained combining basic measures in weighted sum (enhanced measures). In Section 4 we introduce the relative and absolute models, then we review the literature on portfolio optimization problems with real features and classify them according to the type of variables used (relative or absolute models). Section 5 is devoted to solution approaches and computational issues. We survey the main algorithms proposed in the literature for portfolio problems with real features classifying them according to their nature in heuristic and exact solution approaches. Even though the main focus is on mixed integer linear programming (MILP) models, we briefly survey also main solution methods for the mean–variance model with real features. A part of this section will also deal with the important computational issue concerning the solution of large size LP problems including a high number of securities and scenarios. We will discuss recent results from the literature showing how computational efficiency in solving huge LP portfolio problems can be addressed taking advantages from LP duality.

2. Introduction to LP solvable models

The portfolio optimization problem considered in this paper follows the original Markowitz formulation and is based on a single period model of investment. At the beginning of a period, an investor allocates the capital among various securities, thus assigning a nonnegative weight (share of the capital) to each security. During the investment period, a security generates a random rate of return. This results in a change of capital invested (observed at the end of the period) which is measured by the weighted average of the individual rates of return.

Let $J = \{1, 2, \ldots, n\}$ denote a set of securities considered for an investment. For each security $j \in J$, its rate of return is represented by a random variable R_j with a given mean $\mu_j = E\ \{R_j\}$. Further, let $\mathbf{x} = (x_j)_{j=1,\ldots,n}$ denote a vector of decision variables x_j expressing the weights defining a portfolio. To represent a portfolio, the weights must satisfy a set of constraints. The basic set of constraints is defined by a requirement that the weights must sum to one, i.e. $\sum_{j=1}^{n} x_j = 1$ and $x_j \geqslant 0$ for $j = 1, \ldots, n$. An investor usually needs to consider some other requirements expressed as a set of additional side constraints. Most of them can be expressed as linear equations and inequalities. We will assume that the basic set of portfolios Q is a general LP feasible set given in a canonical form as a system of linear equations with nonnegative variables. Although, in farther sections we show that taking into account real features such as transaction costs, minimum transaction lots, cardinality constraints, thresholds on maximum or minimum investments in most cases requires the introduction of integer and binary variables into the LP structure.

Each portfolio $\mathbf{x}$ defines a corresponding random variable $R_{\mathbf{x}} = \sum_{j=1}^{n} R_j x_j$ that represents a portfolio rate of return. The mean rate of return for portfolio $\mathbf{x}$ is given as: $\mu(\mathbf{x}) = \mathbb{E}\{R_{\mathbf{x}}\} = \sum_{j=1}^{n} \mu_j x_j$. Following Markowitz (1952), the portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem

$$\max\{[\mu(\mathbf{x}), -\varrho(\mathbf{x})] : \quad \mathbf{x} \in Q\} \tag{1}$$

where the mean $\mu(\mathbf{x})$ is maximized and the risk measure $\varrho(\mathbf{x})$ is minimized. A feasible portfolio $\mathbf{x}^0 \in Q$ is called the efficient solution of problem (1) or the μ/ϱ-*efficient* portfolio if there is no $\mathbf{x} \in Q$ such that $\mu(\mathbf{x}) \geqslant \mu(\mathbf{x}^0)$ and $\varrho(\mathbf{x}) \leqslant \varrho(\mathbf{x}^0)$ with at least one inequality strict.

In the original Markowitz model (Markowitz, 1952) the risk is measured by the standard deviation or variance: $\sigma^2(\mathbf{x}) = \mathbb{E}\{(\mu(\mathbf{x}) - R_{\mathbf{x}})^2\}$. Several other risk measures have been later considered thus creating the entire family of mean-risk (Markowitz type) models (cf. Mansini, Ogryczak, & Speranza (2003b)). We focus our analysis on the class of Markowitz-type mean-risk models where risk measures, similar to the standard deviation, are shift independent dispersion parameters. Thus, they are equal to 0 in the case of a risk free portfolio and take positive values for any risky portfolio. Moreover, in order to model possible advantages of a portfolio diversification, risk measure $\varrho(\mathbf{x})$ must be a convex function of $\mathbf{x}$.