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Continuous Optimization

A geometric characterisation of the quadratic min-power centre ☆

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ABSTRACT

For a given set of nodes in the plane the min-power centre is a point such that the cost of the star centred at this point and spanning all nodes is minimised. The cost of the star is defined as the sum of the costs of its nodes, where the cost of a node is an increasing function of the length of its longest incident edge. The min-power centre problem provides a model for optimally locating a cluster-head amongst a set of radio transmitters, however, the problem can also be formulated within a bicriteria location model involving the 1-centre and a generalised Fermat-Weber point, making it suitable for a variety of facility location problems. We use farthest point Voronoi diagrams and Delaunay triangulations to provide a complete geometric description of the min-power centre of a finite set of nodes in the Euclidean plane when cost is a quadratic function. This leads to a new linear-time algorithm for its construction when the convex hull of the nodes is given. We also provide an upper bound for the performance of the centroid as an approximation to the quadratic min-power centre. Finally, we briefly describe the relationship between solutions under quadratic cost and solutions under more general cost functions.

1. Introduction

One of the most important problems in the optimal design of wireless ad hoc radio networks is that of power minimisation. This is true during the physical design phase and when designing efficient routing protocols (Montemanni, Leggieri, & Triki, 2008; Yuan & Haugland, 2012; Zhu, Huang, Chen, & Wang, 2012). The most appropriate fundamental model in both cases is the *power efficient range assignment problem*, where a communication range r_i is assigned to each transmitter x_i such the resultant network is connected and that total power $\sum r_i^\alpha$ is minimised (see Althaus et al., 2006). The exponent α is called the *path loss exponent* and most commonly takes a value between 2 and 4, with $\alpha = 2$ corresponding to transmission in free-space. In this paper we study a type of power efficient range assignment problem which allows for the introduction of a single additional transmitter in the plane. We show that this formulation is related to a continuous version of the *cent-dian* problem (Colebrook & Sicilia, 2007) from location analysis, and present a geometric method (based on farthest point Voronoi diagrams) of constructing the optimal solution when $\alpha = 2$.

The classical range assignment problem, which does not include the option of introducing new transmitters, is a type of *disk covering problem*, where the centres of the disks are given nodes, the radii (r_i) of the disks are transmission ranges, and the directed graph induced by the disks must satisfy a given connectivity constraint (for instance strong connectivity, biconnectivity, etc.) whilst minimising $\sum r_i^\alpha$; see Fig. 1(a), where the graph drawn is not necessarily optimal. Observe that graphs that result from some assignment of ranges are similar to *unit-disk* graphs, except that the disks do not all have the same radius in our case. For any $\alpha > 1$ the range assignment problem is NP-hard, even in the case when only 1-connectivity is required of the resultant network (Fuchs, 2008).

The idea of extending the classical range assignment problem by allowing the introduction of additional transmitters is justified because, during the design or maintenance of ad hoc radio networks, it is often pertinent to introduce relays or *cluster-heads* for the processing of aggregated data and for the improved routing efficiency that takes place in such hierarchical structures (see Dhanaraj & Murthy, 2007; Paul & Matin, 2011; Shi, Jia, & Hai tao, 2009). Solving the range assignment problem whilst allowing for the introduction of a bounded number of additional nodes anywhere in the plane constitutes a very general and highly applicable geometric network problem, which has only been solved in certain restricted settings (see for instance Brazil, Ras, & Thomas, 2012; Brazil, Ras, & Thomas, 2010). Since the optimal locations of the cluster-heads must be found, as well as the optimal assignment of ranges on the complete set of nodes, this so called *geometric range assignment problem* is at least as difficult as the classical range assignment problem. Note that this problem is a type of *continuous location* problem, since the cluster-heads are free to be located anywhere in the plane.

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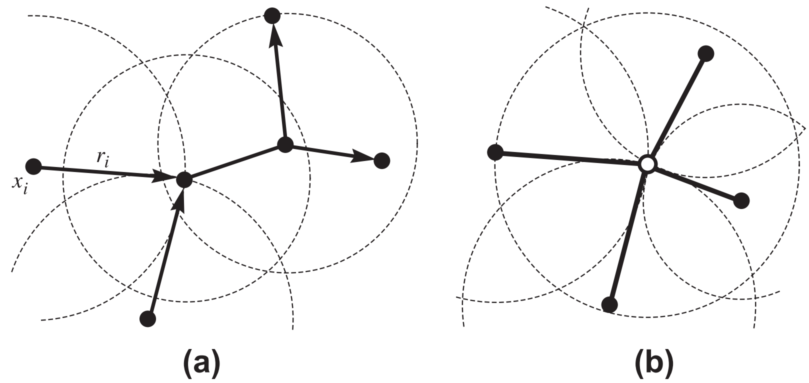

Fig. 1. Examples of range assignment-induced graphs (or disk graphs). Circles centred at nodes represent the transmission distances of the respective nodes. Bidirected edges are drawn without arrows.

This paper considers the problem of optimally locating a *single* cluster-head amongst a given set of transmitters, where each transmitter can send and receive data *directly* to and from the cluster-head. Not only is this an interesting and applicable model in it own right, but it is also a necessary first step in understanding the local structure of optimal networks with multiple cluster-heads. The graph induced by the assignment of ranges, in this case, contains an undirected star with the cluster head as its centre and the complete set of transmitters as its leaves; see for instance Fig. 1(b), where the white node represents a cluster-head.

In more formal terms we denote the given finite set of transmitters by $X \subset \mathbb{R}^2$ and the cluster-head by $s \in \mathbb{R}^2$. The *power* of any $x \in X$ is $P_x = \|s - x\|^\alpha$ and the *power* of s is $P_s = \max\{\|s - x\|^\alpha : x \in X\}$, where $\|\cdot\|$ is the Euclidean norm. The *total power* of the system is denoted by $P(s) = P_s + \sum_{x \in X} P_x$, and a *min-power centre* of X is a point s^* which minimises $P(s)$. The *min-power problem*, which is the problem we address in this paper, seeks to locate a min-power centre of a given set X in the plane. Minimising only P_s is clearly equivalent to the 1-*centre problem*, i.e., the problem of finding the centre of a minimum spanning circle for X. Minimising only $\sum P_x$ is a *generalised Fermat-Weber* problem (Brimberg & Love, 1999), which becomes the classical Fermat-Weber problem when $\alpha = 1$ and the problem of constructing the *centroid* when $\alpha = 2$.

A concept in facility location that is related to the min-power problem is the computation of the *centre-median* (or *cent-dian*) of a finite set of points (Colebrook & Sicilia, 2007); although, strictly speaking, the cent-dian is only defined for $\alpha = 1$. The cent-dian problem requires one to find the Pareto-optimal solutions to the vector function $\Phi(s) = (P_s, \sum P_x)$, which is equivalent to finding the optimal value of $\lambda P_s + (1 - \lambda) \sum P_x$ for every $\lambda \in [0,1]$ (see Duin & Volgenant, 2012; Fernandez, Nickel, Puerto, & Rodríguez-Chía, 2001). The min-power problem may be viewed as a type of cent-dian problem, since it results by setting $\lambda = 1/2$ and by allowing other values of α besides $\alpha = 1$. The cent-dian problem (with $\alpha = 1$) has been considered in the rectilinear plane (McGinnis & White, 1978), and for $\alpha = 2$ in the Euclidean plane (Ohsawa, 1999). Besides optimally locating a cluster-head amongst given transmitters, the cent-dian also has another application in wireless ad hoc networks, namely, finding a so called *core* node (Dvir & Segal, 2010). Bicriteria models such as the cent-dian have been described as seeking a balance between the antagonistic objectives of *efficiency* (i.e., the minisum component) and *equity* (i.e., the minimax component).

There are numerical methods, for instance the sub-gradient method, that optimally locate a min-power centre to within any finite precision. However, structural results for the min-power centre problem, of the type described in this paper, are necessary for optimally constructing more complex geometric range assignment networks (which is the overarching goal of our research). This fact is particularly manifest in the design of algorithmic *pruning* modules, where one develops strategies based on properties of locally optimal structures for eliminating suboptimal network topologies from the exponential set of possible topologies. The ultimate benefits of good pruning modules has been demonstrated a number of times for problems similar to the geometric range assignment problem (Brazil et al., 2012; Brazil et al., 2010; Warme, Winter, & Zachariasen, 2000).

In this paper we mostly focus on the quadratic case, $\alpha = 2$, and develop a complete geometric description of the solution in terms of farthest point Voronoi diagrams and Delaunay triangulations. In terms of cluster-head placement the $\alpha = 2$ assumption means that radio transmission takes place in free space, that is, in an ideal medium with zero resistance. Path loss exponents close to 2 frequently occur in real-world wireless radio network scenarios. This is true for transmission in mediums of low resistance, and in mediums of higher resistance when there is a degree of *beam forming* (constructive interference) (Karl & Willig, 2007). The quadratic case also applies to certain classical facility location problems, including the location of emergency facilities such as hospitals and fire stations (Fernandez et al., 2001; Ohsawa, 1999; Puerto, Rodríguez-Chía, & Tamir, 2010). Furthermore, as demonstrated in the final section, it is anticipated that theoretical developments in the $\alpha = 2$ case will lead to solutions and approximations for other $\alpha > 1$.

Single-facility location problems under very general norms and convex cost functions have been studied by Durier (1995) and by Durier and Michelot (1985, 1994). Some of these formulations include the min-power problem as a special case, however, a constructive method of producing the min-power centre does not directly follow from their work. The principle contribution of the research presented by Durier and Michelot in these papers involves a description of the set of solutions to generalised Fermat-Weber problems, which is a feasible and interesting endeavour when the objective cost function is not strictly convex (as is the case with the min-power problem). Nickel, Puerto, and Rodríguez-Chía (2003) generalise this approach even further by allowing each given point (facility) to be replaced by a set of points, where an optimal centre is required to "serve" at least one point from each given set. A stochastic version of generalised single-facility location problems has also been studied (see Puerto & Rodríguez-Chía (2011)).

In Section 2 we provide definitions and set up a Karush–Kuhn–Tucker formulation of the min-power centre problem and its dual for any $\alpha > 1$. When $\alpha = 2$ the geometric construction of the min-power centre becomes tractable, allowing us to provide a characterisation of the solution in terms of the *farthest point Voronoi diagram* on X, and its dual, the *farthest point Delaunay triangulation*. This characterisation is described in Section 3, where we