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Decision Support

# Efficiency decomposition for general multi-stage systems in data envelopment analysis

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ABSTRACT

Conventional data envelopment analysis (DEA) models only consider the inputs supplied to the system and the outputs produced from the system in measuring efficiency, ignoring the operations of the internal processes. The results thus obtained sometimes are misleading. This paper discusses the efficiency measurement and decomposition of general multi-stage systems, where each stage consumes exogenous inputs and intermediate products (produced from the preceding stage) to produce exogenous outputs and intermediate products (for the succeeding stage to use). A relational model is developed to measure the system and stage efficiencies at the same time. By transforming the system into a series of parallel structures, the system efficiency is decomposed into the product of a modification of the stage efficiencies. Efficiency decomposition enables decision makers to identify the stages that cause the inefficiency of the system, and to effectively improve the performance of the system. An example of an electricity service system is used to explain the idea of efficiency decomposition.



## 1. Introduction

Data envelopment analysis (DEA) is an effective approach for measuring the relative efficiency of a set of decision making units (DMUs) that use multiple inputs to produce multiple outputs. Conventionally, for systems composed of several processes, only the inputs supplied to the system and the outputs produced from the system are considered, ignoring the operations of the internal processes. A result is that, while some processes are inefficient, the system is still evaluated as efficient. In order to produce more appropriate efficiency measures, the operations of the internal processes must be taken into account.

The simplest case of a system with more than one process that has been discussed in the literature is the two-stage system, where two processes are connected in series. Early works on two-stage systems calculated the two process efficiencies separately and independently by treating each stage as a DMU (Charnes et al., 1986; Lovell, Walters, & Wood, 1994; Seiford & Zhu, 1999; Wang, Gopal, & Zionts, 1997). The first attempt of taking the operations of the component processes into account in measuring the system efficiency was probably Färe (1991). Later works include Färe and Whittaker (1995), Färe and Grosskopf (1996), and Löthgren and Tambour (1999). In these studies, the process efficiencies must be calculated separately and independently. It was until Kao and Hwang (2008) that the system and process efficiencies were calculated at the same time, and the relationship that the system efficiency is the product of the process efficiencies was obtained. Based on the same idea, Chen, Cook, Ning, and Zhu (2009) developed a model which is also able to calculate the system and process efficiencies at the same time, in that the former is a weighted average of the latter. The efficiency decomposition discussed in these two studies can be extended to systems of more than two stages. With the efficiency decomposition, the processes that affect the system performance the most are identified. Making improvements in these processes will effectively improve the performance of the overall system.

In the two-stage system, inputs are supplied to the first stage to produce intermediate products for the second stage to convert to final outputs. The characteristics of this type of system are that the first stage does not produce exogenous outputs and the second stage does not consume exogenous inputs. A more general situation is to relax these two restrictions, and several studies on this generalized two-stage systems have been reported (Chen, Chang, Yu, & Hsu, 2012; Chen & Guan, 2012; Kao & Hwang, 2010; Li, Chen, Liang, & Xie, 2012; Premachandra, Zhu, Watson, & Galagedera, 2012). The two-stage system with shared resources (Chen, Du, Sherman, & Zhu, 2010; Golany, Hackman, & Passy, 2006; Yu & Chen, 2011) is a variation of the general two-stage system. The model proposed by Li et al. (2012) is able to measure the system and process efficiencies at the same time, and to decompose the former into a weighted average of the latter. The idea is applicable for general multiple stage systems (Cook, Zhu, Yang, & Bi, 2010). However, the system efficiency was defined as the ratio of the

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aggregate output to the sum of the aggregate input and aggregate intermediate product, which is different from the conventional definition of the ratio of the aggregate output to the aggregate input (without intermediate products). In the current study, we develop a model for measuring the system and process efficiencies at the same time for general multi-stage systems. Different from the studies of Li et al. (2012) and Cook et al. (2010), the system efficiency defined in the current study is the same as the conventional definition of the ratio of the aggregate output to the aggregate input.

Multi-stage systems are special cases of general network systems, and can thus be studied by the models developed for the latter. Specifically, Tone and Tsutsui (2009) applied a slacks-based measure (SBM) approach to measure the process efficiencies, and defined the system efficiency as a weighted arithmetic average of the process efficiencies in an input-oriented model and a weighted harmonic average in an output-oriented one. A non-oriented model was also proposed, in that the system efficiency is defined as a function of the input and output process efficiencies. The weights are pre-determined in these models. In the current study, rather than aggregating the process efficiencies to form the system efficiency through pre-determined weights, we will derive a relationship between the system and process efficiencies measured by the conventional definition.

As far as the structure of the system is concerned, dynamic systems, where the operation of a DMU repeats from period to period and two consecutive periods are connected by carryovers, are also a type of general multi-stage system. Different models have been proposed for measuring the efficiency of these (Chen & van Dalen, 2010; Jaenicke, 2000; Kao, 2013; Nemoto & Goto, 1999; Tone & Tsutsui, 2010). Among these, the model in Kao (2013) is able to decompose the complement of the system efficiency into a linear combination of those of the process efficiencies. However, the results cannot be extended to general multi-stage systems. On the contrary, the properties obtained from multi-stage systems are applicable to dynamic ones. Most significantly, the property that the system efficiency is the product of the modified process efficiencies (Property 4 appeared in Section 4) is not observed in Kao (2013). Besides, the multi-stage model developed in this paper is also able to study series systems with shared inputs, whereas the dynamic model developed in Kao (2013) cannot.

The basic idea of efficiency decomposition in this paper is to transform the general multi-stage system into a series structure, where each stage on the series has a parallel structure. Based on the efficiency decomposition for series and parallel structures, the system efficiency of a general multi-stage system can be expressed as a function of the process efficiencies. The model developed in this paper is applicable not only to standard multi-stage systems, where each stage consumes both endogenous and exogenous inputs and produces both endogenous and exogenous outputs, but also to systems with a structure resembling the general multi-stage one, such as the afore-mentioned dynamic systems and basic multi-stage systems with shared inputs.

In the following sections, the model for measuring the system and process efficiencies of the general multi-stage system is first developed. Efficiency decompositions for the two basic network structures, series and parallel, are then described. After that, the method for decomposing the system efficiency of the general multi-stage system is introduced, and its extensions discussed. Finally, some conclusions are drawn from the discussion.

## 2. General multi-stage system

Let $X_{ij}$, $i = 1,\ldots,m$, and $Y_{rj}$, $r = 1,\ldots,s$, denote the $i$th input and $r$th output of the $j$th DMU, $j = 1,\ldots,n$, respectively. The efficiency of the $k$th DMU under the assumption of constant returns to scale can be measured as (Charnes, Cooper, & Rhodes, 1978):

$$
\begin{aligned}
E_k^{CCR} = \max.\quad & \sum_{r=1}^{s} u_r Y_{rk} \\
\text{s.t.}\quad & \sum_{i=1}^{m} v_i X_{ik} = 1 \\
& \sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} \leqslant 0, \quad j = 1,\ldots,n \\
& u_r,\ v_i \geqslant \varepsilon, \quad r = 1,\ldots,s,\ i = 1,\ldots,m
\end{aligned} \tag{1}
$$

where $u_r$ and $v_i$ are virtual multipliers, and $\varepsilon$ is a small non-Archimedean number imposed to avoid ignoring any factor in calculating efficiency (Charnes & Cooper, 1984). Since in this formulation the operations of the internal processes are not considered, it is referred to as a black-box model to distinguish it from the network one.

The general multi-stage system is a network system composed of a number of processes connected in series, with the structure shown in Fig. 1. In this system, the first process consumes exogenous inputs $X_i^{(1)}$, $i \in I^{(1)}$, supplied from outside of the system, to produce exogenous outputs $Y_r^{(1)}$, $r \in O^{(1)}$, as outputs of the system, and intermediate products $Z_f^{(1)}$, $f \in M^{(1)}$ for the next process to use. In the subsequent processes, each process $p$ applies exogenous inputs $X_i^{(p)}$, $i \in I^{(p)}$ and intermediate products $Z_f^{(p-1)}$, $f \in M^{(p-1)}$, produced by its preceding process, to produce exogenous outputs $Y_r^{(p)}$, $r \in O^{(p)}$ and intermediate products $Z_f^{(p)}$, $f \in M^{(p)}$, for the succeeding process to use. Finally, in process $q$, inputs from outside, $X_i^{(q)}$, $i \in I^{(q)}$, together with intermediate products $Z_f^{(q-1)}$, $f \in M^{(q-1)}$, produced by the preceding process $q-1$, are utilized to produce exogenous outputs $Y_r^{(q)}$, $r \in O^{(q)}$. The sets $I^{(p)}$, $O^{(p)}$, and $M^{(p)}$ contain the indices of the inputs, outputs, and intermediate products for process $p$, respectively.

If the operations of the internal processes are ignored, then the total input supplied to the system is $\sum_{p=1}^{q} X_i^{(p)} = X_i,\ i = 1,\ldots,m$, and the total outputs produced from the system is $\sum_{p=1}^{q} Y_r^{(p)} = Y_r,\ r = 1,\ldots,s$. To measure the system and process efficiencies at the same time for general network systems, Kao (2009a) proposed a relational model which requires the same factor to have the same multiplier associated with it, no matter which process it corresponds to and whether it plays the role of input or output. Similar to the black-box model that requires the aggregate output to be less than or equal to the aggregate input for the system for each DMU, the network model additionally requires the aggregate output to be less than or equal to the aggregate input for each process. The network model for the general multi-stage system of Fig. 1 thus becomes:

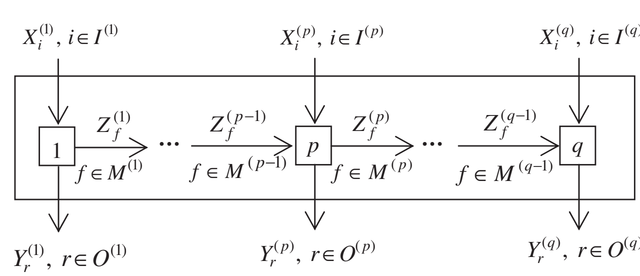


**Fig. 1.** General multi-stage system.