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Decision Support

Selfish routing in public services

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ABSTRACT

It is well observed that individual behaviour can have an effect on the efficiency of queueing systems. The impact of this behaviour on the economic efficiency of public services is considered in this paper where we present results concerning the congestion related implications of decisions made by individuals when choosing between facilities. The work presented has important managerial implications at a public policy level when considering the effect of allowing individuals to choose between providers. We show that in general the introduction of choice in an already inefficient system will not have a negative effect. Introducing choice in a system that copes with demand will have a negative effect.

1. Introduction

What damage do the selfish choices of some afflict onto the welfare of all? The work presented in this paper answers this question in the context of public service systems.

There is a substantial quantity of literature on the subject of equilibrium behaviour of a queueing system where congestion is a factor influencing behaviour [1,5,12,23,31,35,51]. This can be traced back to a series of short communications between Leeman [28,29] and Saaty [44]. This paper builds on this literature by considering the problem of public service systems. For most public services, congestion is a negative aspect of service quality. Examples of this are healthcare systems (waiting lists), transports systems (traffic jams) and/or schools (overcrowding of class rooms).

The degree of central control that should be exercised is a very important question to be considered by governments and/or policy makers. What is the effect of allowing individuals to choose service provider?

Of course, the motivation for the introduction of choice is to create competition in the hope that this would improve overall service quality (an economic evaluation of this point of view can be found in [40]). The aim of the work presented here is to use a game theoretical approach to quantify the effect of removing central control for a given system. For example, consider a situation where the particular providers of a system might have *optimised* their service delivery (perhaps due to competition). In this system if we were to start routing individuals so as to optimise *overall* quality of service, we would (by definition) notice an improvement over a situation where individuals had choice. This impact of choice is the one considered here. The approach proposed is based on a measure called *the price of anarchy* [2,7,8,10,14,18–20,34,41–43,47,52]. First introduced in [25] (a conference paper that has been reprinted in [26]), the price of anarchy is the ratio of the costs of the worst possible Nash equilibrium and the social optimum, and can therefore be interpreted as a measure of the efficiency of a system.

The main contributions of this paper are as follows:

- A novel connection is made: placing choice between public services within the formulation of routing games.
- Theoretical results are obtained as to the effect of demand and worth of service.
- It is shown that in a public service system with an adequate capacity to provide the perceived worth of service, a high price of anarchy is to be expected.
- A numerical approach based on heuristics is proposed to calculate the price of anarchy in a real world setting.
- The above ideas are demonstrated with a large scale real world case study.

The paper is organised as follows: Section 2 will give a brief overview of routing games; Section 3 will interpret choice of public services as a routing game; Section 4 will study a particular model that gives an insightful conclusion as to efficiency of general public service providers under choice; Section 5 looks at an application using hospital performance data for elective knee replacement surgery in Wales; Section 6 makes concluding remarks.

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2. Routing games

This section gives a brief introduction to routing games following very closely [37]. For a thorough overview of routing games the reader is encouraged to view [37,42].

A *non-atomic* routing game (atomic routing games will not be considered in this work), is defined on a network $G = (V, E)$, with vertex set V and edge set E, as well as a set of *source-sink* pairs: $\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}$. These pairs are called *commodities*. The set of all possible (s_i, t_i) paths (for $i \in [k]$) is denoted as $\mathcal{P}_i$. Thus, $\mathcal{P}_i$ denotes the set of all the possible routes that player i may take to get from source s_i to sink t_i. Only networks with $\mathcal{P}_i \neq \emptyset$ for all $i \in [k]$ are considered and $\mathcal{P} = \bigcup_{i \in [k]} \mathcal{P}_i$. The graph G is allowed to have multiple edges and a vertex can participate in multiple source-sink pairs.

The routes taken by traffic are called *flows*, where $f \in \mathbb{R}^{|\mathcal{P}|}_{\geqslant 0}$ denotes a particular flow and f_P is interpreted as the quantity of traffic of commodity i choosing path P for $P \in \mathcal{P}_i$. A flow f is called *feasible* for $r \in \mathbb{R}^k_{\geqslant 0}$ if and only if $\sum_{P \in \mathcal{P}_i} f_P = r_i$ for all $i \in [k]$. Thus, the vector r denotes a prescribed quantity of traffic that must travel from sources to sinks.

What is now needed is some way of differentiating the various paths (indeed some paths may be *better* than others). Each edge $e \in E$ of G has a cost function $c_e : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ and it is assumed that c_e is non-negative, continuous, convex and non-decreasing. Using this, one can quantify the efficiency of a flow f; we define the cost of a flow $C(f)$:

$$C(f) := \sum_{P \in \mathcal{P}} c_P(f) f_p$$

where $c_P(f)$ naturally denotes the cost incurred by the traffic choosing path P. A routing game is then defined by the triple (G, r, c).

Importantly this cost function takes into account the quantity travelling through a particular path. It is immediate to give an equivalent definition:

$$C(f) := \sum_{e \in E} c_e(f) f_e \tag{1}$$

where f_e corresponds to the quantity of traffic using edge e.

Using this we give the following definitions:

Definition 2.1. For the routing game (G, r, c), the flow f^* is an *optimal flow* if and only if f^* minimises C (as given by (1)) over all feasible flows f.

The next definition corresponds to an absence of central control. Note that the term "Wardrop equilibrium" [48] is also used however we choose to use the term "Nash flow" in line with [42].

Definition 2.2. For the routing game (G, r, c), the flow $\tilde{f}$ is a *Nash flow* if and only if for every commodity $i \in [k]$ and every pair of paths P_1, P_2 with $\tilde{f}_{P_1} > 0$ we have:

$$c_{P_1}(\tilde{f}) \leqslant c_{P_2}(\tilde{f})$$

Thus, $\tilde{f}$ is a Nash flow if and only if all used paths have minimum possible cost. This ensures that no user can improve their situation. We now state (without proof) two very powerful results obtained in [4].

Theorem 2.3. *The flow f^* is an optimal flow for (G, r, c) if and only if f^* is a Nash flow for the instance (G, r, c^*) where:*

$$c^*_e(x) = \frac{d}{dx} x c_e(x) = c_e(x) + x \frac{d}{dx} c_e(x) \tag{2}$$

The cost $c^*_e(x)$ is called the *marginal cost* for e. This powerful result shows that mathematically, Nash flows and optimal flows are analogous. The next result simply confirms this.

Theorem 2.4. *The flow $\tilde{f}$ is a Nash flow for (G, r, c) if and only if $\tilde{f}$ minimises Φ where:*

$$\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x) dx \tag{3}$$

The function Φ is called the potential function for (G, r, c). As stated we give these results without proof but encourage the reader to see [37,42].

The object of the work in this paper is to present a measure of the efficiency of a system, the measure we shall use is given by the following definition.

Definition 2.5. For the routing game (G, r, c), the *price of anarchy* denoted by $PoA(G, r, c)$ is given by:

$$PoA(G, r, c) = \frac{C(\tilde{f})}{C(f^*)}$$

Note that the definition taken here differs slightly to the common definition taken as the worst case ratio. The price of anarchy quantifies the inefficiency created by choice. We illustrate these ideas through a famous example, known as, Pigou's example [39].

The network of Fig. 1 corresponds to the routing game, where traffic has a choice of two paths to reach the sink. The upper arc corresponds to a large highway and the travel time is independent (say 1 hour) of the quantity of traffic using that highway. The lower arc corresponds to a much smaller road, that is heavily affected by congestion and the time spent on this road is equivalent to the proportion of traffic that uses it. It is immediate to note that the Nash flow of this game is given by $\tilde{f} = (0, 1)$, since all traffic will go along the smaller road (thus incurring an hour of travel time), in the hope that at least a small quantity of traffic will use the larger road. The cost function for this game is:

$$C(f_1, f_2) = C(1 - x, x) = 1 - x + x^2$$

thus, the optimal flow is $f^* = (\frac{1}{2}, \frac{1}{2})$. It is then straightforward to calculate the price of anarchy: $PoA(G, r, c) = \frac{4}{3}$.

Note that the potential function for this game is:

$$\Phi(f_1, f_2) = \Phi(1 - x, x) = \int_0^{1-x} 1 dx + \int_0^x x dx = 1 - x + \frac{x^2}{2}$$

minimising Φ gives $\tilde{f} = (0, 1)$ as required. For this simple game, calculating the Nash flow does not require using the potential function. However, optimising Φ gives an algorithmic approach. This will be the method used when calculating the price of anarchy for large systems.

In the next section we show how these ideas will be used to measure the efficiency of a public service system.

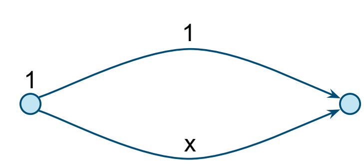

Fig. 1. Pigou's example.