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Discrete Optimization

New lower bounds for the three-dimensional orthogonal bin packing problem ☆,☆☆

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ABSTRACT

In this paper, we consider the three-dimensional orthogonal bin packing problem, which is a generalization of the well-known bin packing problem. We present new lower bounds for the problem from a combinatorial point of view and demonstrate that they theoretically dominate all previous results from the literature. The comparison is also done concerning asymptotic worst-case performance ratios. The new lower bounds can be more efficiently computed in polynomial time. In addition, we study the non-oriented model, which allows items to be rotated.

1. Introduction

The bin packing problem (abbreviated as 1D-BP) is one of the classic NP-hard combinatorial optimization problems. Given a set of n items with positive sizes $v_1, v_2, \ldots, v_n \leqslant B$, the objective is to find a packing in bins of equal capacity B to minimize the number of bins required. The problem finds obvious practical usage in many industrial applications, such as the container loading problem and the cutting stock problem.

The bin packing problem is strongly NP-hard, and it does not admit a $(\frac{3}{2} - \epsilon)$-factor approximation algorithm unless P = NP [16]. On the other hand, Johnson [18] showed that the simple *First Fit* approach can yield a $\frac{17}{10}$-approximation factor, and the *First Fit Decreasing* algorithm can approximate within an asymptotic $\frac{11}{9}$-factor. Subsequently, Fernandez de la Vega and Lueker [15] proposed an asymptotic polynomial time approximation scheme (PTAS), and Karmarkar and Karp [19] presented an improved asymptotic fully PTAS. For further details of approximation algorithms, readers may refer to Coffman, Garey and Johnson's survey [9] and Chapter 9 in [33].

There are many variations of the bin packing problem, such as the strip packing, square packing and rectangular box packing problems. In this paper, we consider the three-dimensional orthogonal bin packing problem (abbreviated as 3D-BP). Given an instance I of n 3D rectangular items $I_1, I_2, \ldots, I_n$, each item I_i is characterized by its width w_i, height h_i, depth d_i and volume $v_i = w_i h_i d_i$. The goal is to determine a non-overlapping axis-parallel packing in identical 3D rectangular bins with width W, height H, depth D and size $B = WHD$ that minimizes the number of bins required. First, we investigate the *oriented model*, which assumes that the orientation of the given items is fixed; that is, the items cannot be rotated and they are packed with each side parallel to the corresponding bin side. The *non-oriented model*, which allows items to be rotated, is also studied.

A considerable amount of research has been devoted to the design and analysis of lower bounds for the bin packing problem [3,6,10,17,20,24,29,32]. Martello and Toth [27,28] and Labbé et al. [22] proposed lower bounds for 1D-BP, and then Martello and Vigo [26] and Martello et al. [25] extended the concept to multi-dimensional models. Fekete and Schepers [13,14] devised lower bounds based on *dual feasible functions*, which we introduce in Section 2; and several related results were presented in [4,8]. Boschetti [1] combined Martello and Toth's work with the above dual feasible functions and proposed the currently best lower bound for 3D-BP in the literature; that is, the lower bound *dominates* all previous 3D-BP results. For two lower bounds L_i and L_j of a minimization problem, L_j is said to *dominate* L_i, denoted by $L_i \leqslant L_j$, if for any instance I, $L_i(I) \leqslant L_j(I)$, where $L(I)$ is the value provided by a lower bound L for an instance I.

In contrast, there have been comparatively few studies on the non-oriented model [1,7,12], especially the three-dimensional model. Dell'Amico et al. [12] presented the first lower bound for the non-oriented model of the two-dimensional orthogonal bin packing problem (abbreviated as 2D-BP). Clautiaux et al. considered different cases of the non-oriented 2D-BP model and proposed a new lower bound [7]; while Boschetti [1] investigated the non-oriented 3D-BP model and presented two new lower bounds.

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In the following sections, we first review the previously proposed lower bounds and integrate the best of them for 1D-BP and 3D-BP to obtain a new lower bound for 3D-BP. Then, we propose a novel lower bound for 3D-BP and show that it dominates all the previous results. We also prove the asymptotic worst-case performance ratios of those results and provide tight examples that can achieve the ratios. Finally, we present a new lower bound for the non-oriented 3D-BP model.

2. Preliminaries

2.1. Lower bounds for 1D-BP

An obvious lower bound for 1D-BP, called the *continuous lower bound*, can be computed as follows:

$$L_0 = \left\lceil \frac{\sum_{i=1}^{n} v_i}{B} \right\rceil$$

It is known that the asymptotic worst-case performance ratio of the continuous lower bound L_0 is $\frac{1}{2}$ for 1D-BP [27,28]. The lower bound can be easily extended to 3D-BP by considering the volume v_i of each item I_i. Martello et al. [25] showed that the worst-case performance ratio of L_0 is $\frac{1}{8}$ for 3D-BP.

Subsequently, the bound was improved by Martello and Toth [28]. Under the new bound, denoted by L_1, the set of items is partitioned into two subsets, one of which contains items that are larger than $B/2$ and the other contains the remainder. For convenience, we define $V(a,b) = \{I_i | a < v_i \leqslant b\}$ and its cardinality as $|V(a,b]|$. Since each item in the first subset needs one bin, at least $|V(B/2,B]|$ bins are required. Only items of size v_i, $p \leqslant v_i \leqslant B$ are considered, where p is an integer with $1 \leqslant p \leqslant B/2$. Hence, a valid lower bound L_1 can be computed if we allow the rest of the items (i.e., the items in $V[p,B/2]$) to be split. The lower bound L_1 for 1D-BP is computed as follows:

$$L_1 = |V(B/2,B]| + \max_{1 \leqslant p \leqslant B/2} \{0, L_1(p), L_1'(p)\}, \text{ where}$$

$$L_1(p) = \left\lceil \frac{\sum_{v_i \in V[p,B-p]} v_i}{B} - |V(B/2, B-p]| \right\rceil \text{and}$$

$$L_1'(p) = \left\lceil \frac{|V[p,B/2]| - \sum_{v_i \in V(B/2,B-p]} \left\lfloor \frac{B-v_i}{p} \right\rfloor}{\left\lfloor \frac{B}{p} \right\rfloor} \right\rceil$$

The key concept of the lower bound L_1 was explained earlier, and in the above formula, $|V(B/2,B]| + \max_{1 \leqslant p \leqslant B/2}\{0, L_1(p)\}$ is specified more precisely. The rounding technique $L_1'(p)$, where $1 \leqslant p \leqslant B/2$, also plays an important role. However, Carlier et al. [4] proved that the *dual feasible function* f_2^p, where $1 \leqslant p \leqslant B/2$, dominates this rounding scheme; therefore, later in the paper, we will apply f_2^p in some cases of our new lower bounds to improve the results reported in the literature. We will introduce the dual feasible functions later. In addition, Martello and Vigo [26] and Martello et al. [25] extended the lower bound L_1 for 1D-BP to the lower bounds of the multi-dimensional models (2D-BP and 3D-BP).

Labbé et al. [22] further improved L_1, denoted as L_2, by partitioning the set of items into three subsets ($V(B/2,B]$, $V(B/3,B/2]$ and $V[p,B/3]$, where $1 \leqslant p \leqslant B/3$) and applying the *First Fit Decreasing* algorithm [9,18,21]. The procedure is implemented as follows. The items in $V(B/2,B]$ are assigned to separate bins as L_1. It may be possible to assign some of the items in $V(B/3,B/2]$ to the open bins, but at most one item in $V(B/3,B/2]$ can fit in any of the open bins. Thus, the open bins are sorted in non-decreasing order based on their residual space, and the items in $V(B/3,B/2]$ are assigned in non-decreasing order of their size. The procedure proves that the items in $V(B/2,B]$ and $V(B/3,B/2]$ can be matched optimally in a pairwise manner. Let K be the subset of items in $V(B/3,B/2]$ that cannot be matched through the above procedure. The items in K can be paired, so at least $\lceil K/2 \rceil$ bins are required. It follows that a valid lower bound L_2 can be derived by allowing the items in $V[p,B/3]$ to be split as follows.

$$L_2 = |V(B/2,B]| + \lceil K/2 \rceil + \max_{1 \leqslant p \leqslant B/3} \{0, L_2(p)\}, \text{ where}$$

$$L_2(p) = \left\lceil \frac{\sum_{v_i \in V[p,B-p]} v_i}{B} - |V(B/2, B-p]| - \lceil K/2 \rceil \right\rceil$$

The lower bound L_2 can be obtained in $O(n)$ time provided that the items are pre-sorted according to their size. Bourjolly and Rebetez [2] and Crainic et al. [11] proved that $L_1 \leqslant L_2$ (excluding the rounding scheme $L_1'(p)$), and that the asymptotic worst-case performance ratio of L_2 for 1D-BP is $\frac{3}{4}$. Note that the primal concept of Labbé et al. cannot be easily extended to a new lower bound L_{m-1} for 1D-BP by partitioning the set of items into m subsets, even by using a brute-force approach. Scholl et al. [31] showed that the lower bound L_2 can be extended by considering the items in $V(B/4,B/3]$, but the process is quite complicated and it does not have any obvious extension.

2.2. The dual feasible functions

A function f:[0,1] → [0,1] is called *dual feasible* if, for any finite set S of non-negative real numbers, the following condition holds:

$$\sum_{x \in S} x \leqslant 1 \Rightarrow \sum_{x \in S} f(x) \leqslant 1$$

The concept of dual feasible functions was first presented by Johnson [18] and subsequently extended by Lueker [23]. Dual feasible functions have been widely studied in the design and analysis of lower bounds for the bin packing problem and its variations. For more detailed information on a variety of dual feasible functions, readers may refer to Clautiaux et al.'s survey [6].

Fekete and Schepers [14] proposed using the concept of dual feasible functions to derive the properties of the lower bounds of the bin packing problem as follows.

Proposition 1 [14]. *Given a dual feasible function f and an instance $I = \{v_1, v_2, \ldots, v_n\}$ of the bin packing problem, a lower bound for the instance $f(I) = \{f(v_1), f(v_2), \ldots, f(v_n)\}$ is also a lower bound for the instance I.*

In this paper, consider two dual feasible functions. The first is the classic dual feasible function for 1D-BP, $f_0^p : [0,B] \to [0,B]$, which is defined as follows [4,14]:

$$f_0^p(x) = \begin{cases} B, & \text{if } x > B-p; \\ x, & \text{if } B-p \geqslant x \geqslant p; \\ 0, & \text{otherwise}, \end{cases}$$

where $1 \leqslant p \leqslant B/2$.

The second is the dual feasible function for 1D-BP, f_2^p, proposed by Carlier et al. [4]. For $1 \leqslant p \leqslant B/2, f_2^p : [0,B] \to [0, 2\lfloor B/p \rfloor]$ is defined as follows:

$$f_2^p(x) = \begin{cases} 2\left(\left\lfloor \frac{B}{p} \right\rfloor - \left\lfloor \frac{B-x}{p} \right\rfloor\right), & \text{if } x > B/2; \\ \left\lfloor \frac{B}{p} \right\rfloor, & \text{if } x = B/2; \\ 2\left\lfloor \frac{x}{p} \right\rfloor, & \text{otherwise}. \end{cases}$$

Clautiaux et al. [8] proved that the above functions f_0^p and f_2^p are *maximal dual feasible functions* (MDFFs) because there are no dual feasible functions larger than them [5,8].

Moreover, by definition, composition and convex combinations of any dual feasible functions are still dual feasible. Thus, the