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Innovative Applications of O.R.

Controlled diffusion processes with Markovian switchings for modeling dynamical engineering systems

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ABSTRACT

In this paper the intrinsic complex nature of engineering systems under control is treated by introducing an approach based on *Controlled Stochastic Differential Equations with Markovian Switchings* (in short *CSDEMS*). Technical conditions for the existence and uniqueness of the solutions of the *CSDEMS* are provided. In this context it is not unusual to deal with non-linear *CSDEMS* that cannot be solved analytically. Therefore, we develop a new two-step, *predictor–corrector method* for finding numerical approximations to solutions of *CSDEMS*. This method utilizes the *Euler–Maruyama method*. An illustrative application to the biochemical engineering area is presented to highlight the usefulness of our approach as a simulation tool.

1. Introduction

Systems of relevant importance in many branches of engineering are controlled dynamical systems evolving in continuous-time over finite time intervals. Under realistic conditions, such systems are in constant interaction with environments of intrinsic variability that could affect seriously their behavior. Mechanical, chemical or biological systems (hereafter referred to as engineering systems) tend to exhibit *hybrid noisy dynamics* that would suggest structural or non-structural changes in their behavior, frequently observed as regimes clearly different from each other during their operation. We are interested in unpredictable and abrupt changes coming from variations in the physical medium properties, operation conditions and control decisions. Such changes could be modeled as *discrete events* driven the evolution (in continuous-time) of that type of systems.

Assuming that the complex nature of an engineering system demands an analysis beyond the scope of deterministic models, a stochastic modeling approach becomes a very powerful tool for describing and analyzing the complexity exhibited by the engineering systems, see for instance [2,10,20,26,40]. For example, we may simulate more realistically (according to different uncertainty scenarios) the system dynamics under different control strategies in order to improve the system performance for reducing operation costs or for increasing the efficiency of such systems.

Two examples are useful to illustrate the ideas pointed out previously. The first one is related to the extraction equipments used in the petroleum industry. The dynamics of these equipments depends on the underground oil compounds and the porous media where they flow. Any variation of such factors modify the flow patterns dynamics of the oil and then, the operation regimes of the extraction processes. This phenomena makes challenging to design adequate control strategies for the oil extraction and shows the potential application of simulation frameworks to deal with such a problem. The second example is related to biological systems (bioreactors) used to produce specialized chemicals. These systems are extremely sensitive to environmental fluctuations (temperature, pH, concentrations of nutrients, etc.) because they depend on biological micro-organisms to carry out the chemical transformations. Thus, if this random facts are not taken into account it will not be possible to set up adequate control strategies for producing high quality products.

The purpose of this paper is twofold: (a) to stablish a general modeling approach for controlled complex engineering systems based on (*CSDEMS*), and (b) to develop an original simulation tool (based on a practical numerical method of solution of (*CSDEMS*)) for exploring and analyzing control strategies which enable us to improve the performance of such systems in a wide range of noise conditions.

It is worth mentioning that in this work the evolution of discrete-time events interacting with the continuous-time dynamics of the engineering systems is modelled as a Markov chain. As a result we have an hybrid system model.

The paper is organized as follows. Section 2 reviews the literature on this topic and presents the framework of our contributions. Section 3 introduces our modeling approach based on *CSDEMS*, including conditions for the existence and uniqueness of the solution. Section 4 presents our numerical approach supported by the

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PC-method. Numerical experiments and simulations as well as an illustrative application to the biochemical engineering field are included in this section. Conclusions and future lines of research are given in Section 5.

2. Literature review

The modeling of complex engineering systems as hybrid stochastic systems is not a novel idea. In this regard, a very complete theoretical support can be found in [7,16] and references therein. A valuable reference concerning applications and advances in this topic is in [8]. On the other hand, there is a great deal of work focused on control problems in dynamical systems, but in our opinion, the contributions in [4,6,9,21] are the ones that stand out.

The use of Stochastic Differential Equations (*SDE*) models in control systems under uncertainty in different fields is mentioned in 2,31]. An specific set of control applications in engineering is included in [11,40]. In this line of thinking, the work in [26] stretches from a theoretical view point the *SDE* framework to include abrupt changes in the system dynamics, bringing about the Stochastic Differential Equations with Markovian Switchings (in short *SDEMS*) models. Valuable application in such a field can be found in [25,27], where side control applications are shown. Nevertheless, as far as we know there is not a systematic approach, based on this settings, that meets a modeling framework and practical methods of solution for solving the equations derived from realistic applications. Our goal in this paper is to provide such a framework through *CSDEMS* models which includes the corresponding theoretical support and a novel numerical method that provides accurate approximate solutions to the *CSDEMS*. As a result, we set up a simulation framework for studying controlled engineering system.

It is very convenient to say that our methodology is not devoted to finding optimal strategies of control. However, it is a simulation-based tool that can be used to select control strategies in dynamical system under uncertainty as it is shown in Section 4.2.2. On the other hand, we can take advantage of this methodology using it together with techniques based on iterative dynamic programming (see [23]). Such techniques seem to be promising to design optimal control strategies in medium-scale and great-scale making-decision problems under uncertainty (see for example [5]).

Regarding the non-linear nature of the problems studied in this work, we deal with non-linear *CSDEMS* whose solution is not reachable by analytic techniques (see for instance [15,18,37]). For that reason it is necessary to explore their solutions through numerical approximations.

The *CSDEMS* approach can potentially be very fruitful for practical applications because numerical methods can be constructed which provide a balance between solution accuracy and ease-of-implementation in terms of computational cost. In this regard, extending the principles in [28,30], we propose a new methodology combining the versatility of the explicit *Euler–Maruyama method* (*EM-method*) and the robustness and stability of the one-step *implicit methods* of solution (see [33]). The idea behind this setting is very intuitive. First we construct an approximation of the solution based on the *EM-method* called the predictor step and secondly we use this value in an implicit scheme to get a refined solution called the corrector step. The complete procedure will be called *predictor–corrector method* (in short *PC-method*).

3. The model and the mathematical support

We consider a noisy engineering system driven by the following *CSDEMS*

$$dx(t) = f(x(t), u(t, x(t), r(t)), r(t))dt + g(x(t), u(t, x(t), r(t)), r(t))dB_t, \tag{1}$$

where $t \in [t_0, T]$; $x(t)$ is the state variable, a random process taking values in $\mathbb{R}^n$; $r(t)$ is the regime and it is modelled by a continuous time Markov chain taking values in a discrete domain S; $u(t,x(t),r(t))$ is a control variable taking values in a compact set $U \subset \mathbb{R}$, which includes possible dependency on the state variables and the regime variable as well (it will be denoted as $u(\cdot)$ for simplicity); and $\{B_t, t \geqslant 0\}$ is a standard Brownian motion defined in $\mathbb{R}^m$.

The dynamics of the system is characterized by the drift function $f: \mathbb{R}^n \times U \times S \to \mathbb{R}^n$ and the diffusion function $g: \mathbb{R}^n \times U \times S \to \mathbb{R}^{n\times m}$, where $\mathbb{R}^{n\times m}$ represents the set of $n \times m$ matrices with real entries. The stochastic process $r(t) \equiv \{r(t), t \geqslant 0\}$is assumed to be a right continuous-time homogeneous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$, with values in a finite state space $S = \{1, 2, \ldots, N\}$ and matrix generator $\Gamma = (\gamma_{ij})_{N\times N}$ defined as follows

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j \end{cases},$$

where $\Delta > 0$ is the step-size fixed on the simulations; and $\gamma_{ij} \geqslant 0$ is the transition rate from the state i to the state j conditioned to

$$\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}. \tag{2}$$

We assume that the Markov chain $r(t)$ is $\mathcal{F}_t$-adapted and independent of the Brownian motion B_t.

Technically, the evolution of the system driven by (1) can be seen as a continuous change between N stochastic differential that evolves according to the dynamics of the Markov chain $r(t)$ regardless of the state of the system and the control actions. Note that the number of such equations coincides with the size of the state space S.

Thereupon we provide the key definitions and concepts of our setting.

Definition 1. Given a control protocol $u(\cdot)$ in $[t_0, T]$, a $\mathbb{R}^n$-valued stochastic process $\{x(t)\}_{t_0 \leqslant t \leqslant T}$ is called a solution of the Eq. (1) if

(i) $\{x(t)\}_{t_0 \leqslant t \leqslant T}$ is continuous and $\mathcal{F}_t$-adapted;
(ii) $f(x(t), u(\cdot), r(t))_{t_0 \leqslant t \leqslant T} in \mathcal{L}^1([t_0, T]; \mathbb{R}^n)$;
(iii) $g(x(t), u(\cdot), r(t))_{t_0 \leqslant t \leqslant T} in \mathcal{L}^2([t_0, T]\mathbb{R}^{n\times m})$.

The solution of (1) is provided by

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), u(\cdot), r(s))ds + \int_{t_0}^{t} g(x(s), u(\cdot), r(s))dB_s \text{ a.s.} \tag{3}$$

A solution $\{x(t)\}_{t_0 \leqslant t \leqslant T}$ is said to be unique if any other solution $\{\bar{x}(t)\}_{t_0 \leqslant t \leqslant T}$ is indistinguishable from the former.

Definition 2. A sample path of the continuous Markov chain $r(t)$ is a right-continuous step function with a finite number of simple jumps on $[t_0, T]$. We consider a sequence $\{\tau_k\}_{k\geqslant 0}$ of stopping times such that

(i) for almost every $\omega \in \Omega$ there is a finite $\bar{k} = \bar{k}(\omega)$ for $t_0 = \tau_0 < \tau_1 < \cdots < \tau_{\bar{k}} = T$ and $\tau_k = T$ if $k > \bar{k}$;
(ii) $r(t)$ is a constant with $r(t) = r(\tau_k)$ on $\tau_k \leqslant t < \tau_{k+1}$.

The conditions of existence and uniqueness of the *CSDEMS* solution are presented in the light of the following Theorem.

Theorem 1. *Assume that functions f(x,u,i) and g(x,u,i) are of the type (ii) and (iii) in Definition 1, they are measurable and satisfy, for some positive constants $\overline{K}$ and K, the following properties*