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Discrete Optimization

On accuracy, robustness and tolerances in vector Boolean optimization

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ABSTRACT

A Boolean programming problem with a finite number of alternatives where initial coefficients (costs) of linear payoff functions are subject to perturbations is considered. We define robust solution as a feasible solution which for a given set of realizations of uncertain parameters guarantees the minimum value of the worst-case relative regret among all feasible solutions. For the Pareto optimality principle, an appropriate definition of the worst-case relative regret is specified. It is shown that this definition is closely related to the concept of accuracy function being recently intensively studied in the literature. We also present the concept of robustness tolerances of a single cost vector. The tolerance is defined as the maximum level of perturbation of the cost vector which does not destroy the solution robustness. We present formulae allowing the calculation of the robustness tolerance obtained for some initial costs. The results are illustrated with several numerical examples.

1. Introduction

While solving practical optimization problems, it is necessary to take into account various kinds of uncertainty due to lack of input data, inadequacy of mathematical models to real processes, rounding off, calculating errors, etc. It is known that in many cases initial data as a link between a reality and a model cannot be defined explicitly. The initial data is defined with a certain error, generally depend on many parameters and require to be specified during the problem solving process. In practice any problem cannot be properly posed and solved without at least implicit use of the results of stability analysis and related issues of parametric analysis. Therefore widespread use of discrete optimization models in the last decades inspired many specialists to investigate various aspects of ill-posed problems theory and, in particular, the stability issues.

The implications of enhanced optimization methods have in some areas been lead to the situation that optimal or near-optimal solutions have become "too good". For example, in design of a communication network, a network configuration can now be made so good (with respect to the original objective optimization) that there is hardly any possibility left in the network to accommodate for potential disruptions and possible contingency in terms of e.g. routing delays. Similar problems are faced nowadays in many other areas where deterministic models do not properly reflect possible uncertainty of input parameters. In practice, it usually leads to undesirable situations where optimality (sometimes even feasibility) of solutions is very sensitive to some possible realizations of problem parameters. Thus, chasing for solution optimality, we lose its robustness and vice versa.

As a consequence, two lines of research within the operations research and mathematical optimization community have been initiated:

- Post-optimal and parametric analysis investigate how an optimal solution found behave in response to initial data (problem parameters) changes. A general sensitivity and stability analysis methodology is used based on analyzing the properties of the point-to-set mapping which specifies the optimality principle of the problem. Such research methods have been studied in great detail and covered e.g. in the literature on optimization problems with a continuous set of feasible solutions. Numerous articles are devoted to analysis of conditions when a problem solution possesses a certain property of invariance to the problem parameters perturbations (see, e.g. [10,31,37,38]).
- Robust optimization – instead of producing an optimal solution for a normal situation, which is described by deterministic models but rarely occurs in practice, and where recovery to optimality can be complicated, the aim is to produce solutions that optimize an additionally constructed objective. The objective must assure that the optimal solution will remain feasible under worst case realization of uncertain problem input parameters. Worst-case optimization is also known as robust optimization, and optimal solutions of worst case optimization are often referred to as robust solutions (see e.g. [15]).

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The main drawback of all classical single objective models is that they do not take into account the real multiple criteria nature of real-life problems. It is well-known that under multiobjective framework a solution which is optimal with respect to one single objective might be arbitrarily bad with respect to the others and thus will be unacceptable for a decision maker. Thus, many problems arising in optimization, management and decision making should be ultimately considered under multicriteria framework due to existing of several conflicting goals or interests. Therefore recent interest of applied mathematicians and operations research scientists in multicriteria optimization problems keeps very high. It is confirmed by the intensive publishing activity (see e.g. monographs [8,23,36] and bibliography [9]).

The main difficulty while studying stability of discrete optimization problems is discrete models complexity, because even small changes of initial data make a model behave in an unpredictable manner. There are a lot of papers (see e.g. [4,11,12,16,33–35]) devoted to analysis of scalar and vector (multicriteria) discrete optimization problems sensitivity to parameters perturbations.

The present work continues investigations of different aspects of sensitivity analysis for different types of discrete optimization problems with various partial criteria and optimality principles (see e.g. [5–7,20,22,27,28]). We consider a multiobjective Boolean linear programming problem with a finite number of alternatives in which initial coefficients (costs) of linear payoff functions are subject to perturbations. We define robust solution as a feasible solution which for a given set of realizations of uncertain parameters guarantees the minimum value of the worst-case relative regret among all feasible solutions. For the Pareto optimality principle, an appropriate definition of the worst-case relative regret is specified. We show that this definition is closely related to the concept of accuracy function which has been recently intensively studied in the literature (see e.g. [17,20,28]). We also present the concept of robustness tolerance of a single cost vector, which is defined as the maximum level of perturbation of the cost vector which does not destroy the solution robustness. In this paper we present formulae which allow calculating the robustness tolerances obtained for some initial costs. We illustrate the results with several numerical examples.

The paper is organized as follows. In Section 2 we formulate the problem in details and define basic Pareto optimality principle. In Section 3 we give a short excursus into the topic of robust optimization and define an appropriate robustness measure. Section 4 is devoted to the concept of accuracy function as a tool of post-optimal analysis which is used to describe the behavior of optimal solution under uncertainty of initial problem data. We specify analytical expression to calculating accuracy function for the chosen optimality principle. We also show that accuracy functions can straightforward be used to analyze solution robustness. In Section 5, we focus on analyzing the case when only one vector cost is uncertain. We present formulae which allow calculating the robustness tolerances. The theoretical results presented in Section 5 are illustrated with numerical examples given in Section 6. Some concluding remarks and open problems are summarized in Section 7.

2. Problem formulation

We consider a generic programming problem of dimension $m \geqslant 2$ with a finite set of alternatives (feasible solutions) encoded by means of Boolean variables x_j, $j \in N_m = \{1, 2, \ldots, m\}$. The set of all feasible solutions X is generally defined as a subset of the Cartesian product over all sets of possible realizations of the decision variables

$$X \subset \prod_{j \in N_m} X_j = \{0,1\}^m.$$

Observe that now – formally – X is a subset of the set of all binary m-tuples. We also assume that there exists at least one j with $x_j = 1$. Thus, $0_{(m)} = (0, 0, \ldots, 0)^T \notin X$. A vector of objective functions:

$$p(C,x) = (p_1(C,x), \ldots, p_n(C,x))^T$$

consists of individual (partial) objectives $p_i(C, x)$, which are defined as linear functions on the set of solutions X:

$$p_i(C,x) = C_i x.$$

Here for every $i \in N_n$, C_i is ith row of matrix $C = [c_{ij}] \in \mathbf{R}_+^{n \times m}$, $x = (x_1, x_2, \ldots, x_m)^T, x_j \in X_j, j \in N_m$. Note that for each $i \in N_n$, the individual objective $p_i(C, x)$ depends on solution x, that is on realization of all the decision variables x_j, $j \in N_m$. Thus, an image set $PP(C, X)$ is the following:

$$PP(C,X) = \{p(C,x) : x \in X\}.$$

The problem consists in simultaneous minimization of the individual objective functions given some original cost matrix C. We will call any such problem a problem with the matrix C. Here we give a short example illustrating the real situation where this particular model can appear. Assume we have m households located in different areas. The households decide about making a joint pipeline or some other sort of shared resource through the areas. Every household has own costs as well as costs related to the participation of the other householders in the project. There are also some restrictions which prohibits certain combination of households to take part in the project simultaneously. Then this situation can be interpreted as the combinatorial model which was described above. It is clear that many well-known combinatorial optimization problems such as shortest path, minimum spanning tree, traveling salesman and network flow problems, can generally fit the combinational model considered.

Now we formulate a classical definition of optimal principle used within multiple objective environment. A solution $x^* \in X$ is called **Pareto optimal** (see e.g. [30]) in the problem with matrix C if there exists no solution $x \in X$ such that $p_i(C,x) \leqslant p_i(C,x^*)$ for all $i \in N_n$, and $p_i(C, x) < p_i(C, x^*)$ for at least one $i \in N_n$. For the problem with matrix C, denote $P^n(C)$ the set of Pareto optimal solutions.

3. Robust deviation

One of the most interesting branches of combinatorial optimization and mathematical programming that has emerged over the past 20–30 years is robust optimization. Since the early 1970s there has been an increasing interest in the use of robust optimization models. The theory of robustness deals with uncertainty of problem parameters. The presence of such parameters in optimization models is caused by inaccuracy of initial data, non-adequacy of models to real processes, errors of numerical methods, errors of rounding off and other factors. So it appears to be important to identify classes of models and their solutions which play against the worst-case (in some sense) realization of input parameters. It is commonly accepted fact nowadays that any optimization problem arising in practice can hardly be adequately formulated and solved without usage of results of the theory of robustness.

Authors of most papers devoted to robust optimization attempt to answer to the following closely related questions: How can one represent uncertainty? What is a robust solution? What could be a proper robustness measure? How to calculate robust solutions? How to interpret worst case realization under uncertainty? and many others. Multiple various research approaches were originated when those questions were scrutinized under different frameworks. Bibliographical analysis provides us with a list of contributors who proposed several main avenues in the theory of robustness: