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Decision Support

A heuristic method to rectify intransitive judgments in pairwise comparison matrices

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ABSTRACT

This paper investigates the effects of intransitive judgments on the consistency of pairwise comparison matrices. Statistical evidence regarding the occurrence of intransitive judgements in pairwise matrices of acceptable consistency is gathered by using a Monte–Carlo simulation, which confirms that relatively high percentage of comparison matrices, satisfying Saaty's CR criterion are ordinally inconsistent. It is also shown that ordinal inconsistency does not necessarily decrease in the group aggregation process, in contrast with cardinal inconsistency. A heuristic algorithm is proposed to improve ordinal consistency by identifying and eliminating intransitivities in pairwise comparison matrices. The proposed algorithm generates near-optimal solutions and outperforms other tested approaches with respect to computation time.

1. Introduction

The pairwise comparisons method, proposed by Thurstone (1927), is often used as an intermediate step in multi-criteria decision making, when the decision maker (DM) is unable to directly assign criteria weights or scores of alternatives. The preference elicitation in two widely used decision making techniques, namely the analytic hierarchy process (Saaty, 1980) and the PROMETHEE methods (Brans and Mareschal, 2005), is based on pairwise comparisons. Pairwise comparisons are also used in voting systems (Merrill, 1984), and multi-agent AI systems (Zhang et al., 2008).

When DMs are presented with a number of elements (criteria or alternatives) which have to be ranked with respect to a preference scale, it is assumed that they can compare each pair of elements and provide an ordinal preference judgement whether an element is preferred to another one (*preference dominance*) or both elements are equally preferred (*preference equivalence*). In the pairwise comparison prioritisation process it is also assumed that DMs are able to express the strength of their preferences by providing additional cardinal information. However, as DMs are often biased in their subjective comparisons, some level of inconsistency of their preference judgements may exist.

If we have three comparison elements *A*, *B* and *C*, *ordinal consistency* (transitivity) means that if *A* is preferred to *B* and *B* is preferred to *C*, then *A* must be preferred to *C*. *Cardinal consistency*, which is a much stronger requirement than ordinal consistency, states that if *A* is preferred to *B* p times, and *B* is preferred to *C* q times, then *A* should be preferred to *C* p^*q times. Obviously, if the DM is cardinally consistent, he/she is ordinally consistent as well. However, ordinal consistency does not imply cardinal consistency.

Pairwise comparison is a vital part of the prioritisation procedure in the analytic hierarchy process (AHP), which provides a comprehensive and rational framework in which to structure a decision problem. In the AHP, pairwise judgements are structured in a pairwise comparison matrix (PCM) and a prioritisation procedure is applied to derive a corresponding priority vector (Choo and Wedley, 2004). If the comparison judgements are cardinally consistent then the constructed PCM is also consistent and all prioritisation methods give the same result. However, in the case of ordinally or cardinally inconsistent judgements, different prioritisation methods derive different priority vectors.

Generally, if the PCMs are ordinally consistent, most prioritisation methods derive priorities having the same ranking, only with different intensities. If, however, the matrices are ordinally inconsistent (intransitive), there exists no priority vector which satisfies all contradictory preferences. Therefore, different prioritisation methods provide different ordinal rankings that partially correspond to the ordinal comparison judgements.

The AHP allows a certain level of inconsistency of the PCM, which is measured by the consistency ratio (*CR*) (Saaty, 1980). *CR* is calculated by using the largest eigenvalue of the comparison

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matrix, which is a global characteristic of that matrix. When $CR < 0.1$, the inconsistency is deemed to be relatively small and the matrix is said to be of *acceptable inconsistency*.

In an attempt to eliminate the drawbacks of the *CR* index, such as the inability of the Eigenvalue to identify inconsistent judgements and the arbitrary threshold value of 0.1, Koczkodaj (1993) proposed a new consistency measure *CM*, based on the properties of basic comparison matrices of 3rd order. The *CM* is defined as a relative distance to the nearest consistent comparison matrix. However, similarly to the *CR* index, the *CM* cannot measure the ordinal inconsistency of the comparison judgements.

In order to measure ordinal inconsistency, Kendall (1955) introduced an ordinal *Coefficient of Consistence* for pairwise comparison with no preference equivalencies between elements. Jensen and Hicks (1993) extended Kendall's work to general reciprocal comparison matrices and proposed an ordinal *consistency index* ζ. They also discussed the use of ordinal consistency indexes in AHP and concluded that ordinal indexes cannot eliminate the need for cardinal consistency measures in the AHP method.

The AHP does not require transitivity of DM's judgements. Saaty's *CR* index measures the cardinal inconsistency of the judgements but does not capture their ordinal inconsistency (this is also true for other cardinal indexes, such as CM). Ordinal inconsistency always implies cardinal inconsistency, however, the converse does not hold. Generally, if the comparison judgements and the corresponding PCM are ordinally inconsistent, the value of CR is above the threshold of 0.1, therefore, the AHP implicitly presumes that satisfying the *CR* test may significantly reduce the chances of ordinal inconsistency. However, there are examples in the literature where matrices that satisfy the *CR* criterion can also be ordinarily inconsistent (Jensen and Hicks, 1993; Kwiesielewicz and van Uden, 2004).

The first objective of this paper is to investigate the effects of intransitive judgments on the consistency of PCMs. The statistical significance of the hypothesis that matrices which satisfy the cardinal consistency test (*acceptable matrices*, in AHP terms) are also ordinally consistent is tested by Monte–Carlo simulation. The results provide large-scale confirmation of the observation of other authors that the *CR* test does not guarantee judgments to be ordinally consistent.

The problem of intransitive judgments in the context of group decision making is also investigated. Although, the *CR* of the aggregated PCM is smaller than the largest individual *CR* (Escobar et al., 2004), it is shown here that the aggregation of ordinally inconsistent PCMs may lead to new intransitivities, which do not exist in the initial individual matrices, and consequently, the group ordinal inconsistency may increase.

These observations lead to the second objective of the paper, which is to propose a procedure to improve the overall consistency of the judgements by detecting and modifying inconsistent ordinal judgements in PCMs. Using a graph-theoretic approach, the removal of intransitivities is formulated as an optimisation problem and a sub-optimal heuristic algorithm is proposed. The heuristic algorithm achieves almost identical results to the optimisation algorithms; however, it is simpler, than the numerical optimisation methods and more efficient from a computational viewpoint. This is demonstrated by comparing its performance to different optimisation methods, such as enumeration and integer linear programming.

The paper is organised as follows: the prioritisation problem is formulated in Section 2; Section 3 discusses ordinal and cardinal inconsistency in PCMs and illustrates the problem of priority derivation from intransitive matrices; Section 4 explains the Monte–Carlo simulations undertaken to gather statistical evidence regarding intransitivity in PCMs; the importance of ordinal consistency in group decision making is discussed in Section 5, and an example to demonstrate how intransitive individual judgments adversely affect the aggregated PCM is given; Section 6 discusses the rectification of intransitive judgments as a graph-theoretic minimum feedback edge-set problem and proposes a heuristic algorithm to detect and correct the most intransitive judgments; Section 7 presents comparison results; and the final section summarises the paper.

2. Prioritisation by pairwise comparisons

Consider a prioritization of n elements $E_1, E_2, \ldots, E_n$. In multi-criteria decision making these elements could be either criteria or alternatives. The DM assesses the relative importance of any two elements E_i and E_j by providing a comparison judgment a_{ij}, specifying by how much E_i is preferred/not preferred to E_j. If the element E_i is preferred to E_j then $a_{ij} > 1$, if the elements are equally preferred, then $a_{ij} = 1$ and if E_j is preferred to E_i then $a_{ij} < 1$.

Each set of pairwise comparisons with n elements requires $n(n-1)$ judgments, however, the number of pairwise comparisons is reduced to $n(n-1)/2$, due to the reciprocal property $a_{ij} = 1/a_{ji}$.

The AHP method structures the comparison judgements in a positive reciprocal PCM such that:

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \cdots & a_{2n} \\ 1/a_{13} & 1/a_{23} & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & \cdots & 1 \end{bmatrix}.$$

We may suppose that there exist a preference vector $r = (r_1, r_2, \ldots, r_n)^T$ such that r_i represents the preference intensity of E_i, $i = 1, 2, \ldots, n$. However, the preference vector r is unknown to the DM and must be estimated. The prioritisation problem is to determine a priority vector $w = (w_1, w_2, \ldots, w_n)^T$ from A, which estimates the unknown preference vector $r = (r_1, r_2, \ldots, r_n)^T$.

There are many prioritisation methods that can be applied to derive a priority vector from PCM. The most widely used methods are the eigenvector method (EV), the direct least squares (DLS), the logarithmic least squares (LLS), the logarithmic lease absolute value (LLAV), the logarithmic absolute error (LAE) and the weighted least squares (WLS). An excellent summary of existing prioritisation methods is given in Choo and Wedley (2004), where more than 20 different methods are analysed and numerically compared. It is shown that in the case of error-free (consistent) judgements, all prioritisation methods give equal results, however, the results are different when the PCMs are inconsistent.

3. Consistency of pairwise comparisons judgements

3.1. Cardinal consistency

The issue of consistency in pairwise comparisons has been discussed by many authors (e.g. Saaty (1980), Jensen and Hicks (1993), Koczkodaj (1993), Kwiesielewicz and van Uden (2004), Hartvigsen (2005), Li and Ma, 2007).

The judgments of DMs are *cardinally consistent* if the following conditions are met (Saaty, 1980):

- $a_{ij} = 1/a_{ji}$ for all i and j;
- $a_{ij} = a_{ik}{}^{*}a_{kj}$, where $j > k > i$.

When the DM is perfectly consistent in his/her judgments, then the priority vector $w = (w_1, w_2, \ldots, w_n)^T$ is exactly the same as the preference vector $r = (r_1, r_2, \ldots, r_n)^T$, and the judgements a_{ij} have