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Short Communication

# A note on replacement policy for a system subject to non-homogeneous pure birth shocks

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ABSTRACT

A system is subject to shocks that arrive according to a non-homogeneous pure birth process. As shocks occur, the system has two types of failures. Type-I failure (minor failure) is removed by a general repair, whereas type-II failure (catastrophic failure) is removed by an unplanned replacement. The occurrence of the failure type is based on some random mechanism which depends on the number of shocks occurred since the last replacement. Under an age replacement policy, a planned (or scheduled) replacement happens whenever an operating system reaches age *T*. The aim of this note is to derive the expected cost functions and characterize the structure of the optimal replacement policy for such a general setting. We show that many previous models are special cases of our general model. A numerical example is presented to show the application of the algorithm and several useful insights.



## 1. Introduction

Many systems deteriorate with age and random repairable failures during operations. In reliability theory, it is worth studying the optimal maintenance or replacement policies to minimize the operating costs and catastrophic breakdown risks. Most of the previous studies focused on the systems subject to shocks according to a non-homogeneous Poisson process (NHPP). The NHPP shock process only depending on system's age is, however, only applicable to systems that are not affected by the number of failures. In this paper, we focus on shocks occurring according to a non-homogeneous pure birth process (NHPBP). In fact, the NHPBP is more appropriate to model the system's deterioration process which not only depends on the system's age, but also the number of shocks. Under the NHPBP shock process, we investigate the maintenance or replacement policies. This general model can be applied in production, insurance, epidemiology and load-sharing systems.

Age replacement policy (ARP) is quite common and easy-to-implement in practice. Under a classical ARP, an operating system is replaced at age *T* or at failure, whichever occurs first (Barlow and Hunter, 1960). Furthermore, maintenance policies including both replacements and minimal repairs have been studied in the literature. Nakagawa (1981) considered a system with two types of failures: type-I failure occurs with probability $q$ and is removed by minimal repair, whereas type-II failure occurs with probability $p(= 1 - q)$ and is removed by replacement. Sheu et al. (1995) proposed an extended replacement policy with general random repair cost and age-dependent minimal repair.

The key issue for the most replacement policies is to develop the expected cost function from which an optimization policy can be determined. There are extensive studies in this line of research. Chen and Savits (1988, 1992) established the expected total $\alpha$-discounted cost for a system under the ARP. Aven and Berman (1986) and Puri and Singh (1986) also considered the ARP optimization problem by the non-monotone marginal cost functions into account. Many works focused on the system subject to shocks which cause system failures. Esary et al. (1973) assumed that shocks occur according to a homogeneous Poisson process. A-Hameed and Proschan (1973) treated the case with shocks following a NHPP. Boland and Proschan (1983) investigated the optimal periodic replacement policy for a system subject to non-homogeneous Poisson shocks and time-independent cost structure. Block et al. (1988) obtained the similar results for the case with time-dependent cost structure. Sheu (1998) considered a more generalized age and block replacement problem under a non-homogeneous Poisson shock process and obtained

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optimal policy existence conditions. A-Hameed and Proschan (1975) presented a pure birth shock model with a nonstationary Markov process: that is given $k$ shocks have occurred in $[0,t]$, the probability of a shock occurring in $(t,t+\Delta)$ is $\lambda_k\lambda(t)\Delta + o(\Delta)$.

In this note, we treat a more general case in which shocks occur according to a non-homogeneous or nonstationary pure birth process. Specifically, we consider a maintenance policy which includes both replacements and repairs (to be described explicitly later). The decision to repair or replace the system at failure depends on the number of shocks suffered since the last replacement. The aim of our study is to examine the structure of the optimal maintenance policy which also generalizes the previous models in the literature. It is worth noting that there were some recent studies using the Markov arrival process (MAP) to model failure process for the case where the major failure occurrence (requiring the replacement) depends on the number of minor repairs (requiring minimal repairs). Montoro-Cazorla et al. (2009) considered systems under shocks whose interarrival times are not independent and modeled the failure process as an MAP. Although there are some advantages in using the MAP process, estimating the transition matrices of the MAP is a quite challenging task in practice and requires extensive data collection. In addition, the analysis based on MAP model is mainly computational approach and it is very hard, if not impossible, to characterize the structure of the optimal policy analytically. Thus we are motivated to use the NHPBP to model the system failure process depending on both age and number of failures.

The rest of this note is organized as follows. Section 2 presents the model formulation and develops the expected cost functions. Section 3 focuses on the optimization of the maintenance policy. Some special cases are discussed in Section 4. Section 5 develops an algorithm for determining to optimal replacement schedule, and a computational example is given to illustrate the application of the algorithm. Finally, Section 6 concludes.

## 2. Model formulation and cost functions

### 2.1. The shock model

Consider a system subject to shocks which occur according to a non-homogeneous or nonstationary pure birth process defined below.

**Definition 1.** If a counting process $\{N(t):\ t \geqslant 0\}$ is a non-homogeneous continuous time Markov process with following conditions:

(i) $N(0) = 0$,
(ii) $P\{N(t+h) - N(t) = 1 | N(t) = k\} = \lambda_k(t) + o(h)$,
(iii) $P\{N(t+h) - N(t) \geqslant 2 | N(t) = k\} = o(h)$,
(iv) the process has independent increments,

then the process is called a non-homogeneous or nonstationary pure birth process (denoted by NHPBP or NSPBP) with the intensity function $\{\lambda_k(t),\ k = 0,1,2,\ldots\}$.

**Remark 1.** If $\lambda_k(t) = \lambda(t)$ for $k = 0,1,2,\ldots$, then the NHPBP is reduced to a non-homogeneous Poisson process (NHPP). If $\lambda_k(t) = \lambda_k\lambda(t)$, it becomes the case considered by A-Hameed and Proschan (1975). If $\lambda_k(t) = k\lambda(t)$, it can be considered as a generalized Yule birth process.

**Remark 2.** There are many studies on the effectiveness of preventive maintenance (PM) for a maintainable system. Most of them model the hazard rates of maintained systems after PM interventions. Nguyen and Murthy (1981) propose that $\lambda_k(t)$ is the hazard rate of the system after the $k$th PM intervention (the initial hazard function is $\lambda(t) = \lambda_0(t)$). According to a taxonomy by Nakagawa (1988) and Lin et al. (2000), existing PM models are categorized into three groups: (i) Hazard rate model if $\lambda_k(t) = a_{k-1}\lambda_{k-1}(t) = \left(\prod_{j=0}^{k-1} a_j\right)\lambda(t)$, (ii) Age reduction model if $\lambda_k(t) = \lambda(b_{k-1}y_{k-1} + t)$, and (iii) Hybrid model if $\lambda_k(t) = \left(\prod_{j=0}^{k-1} a_j\right)\lambda(b_{k-1}y_{k-1} + t)$, where $a_k, b_k$ are non-negative adjustment factors, and $y_k$ is the effective age of the system just prior to the $k$th PM for $k = 1,2,\ldots$ Wu and Zuo (2010) extend the existing PM models to three new ones: (i) Linear model if $\lambda_k(t) = a_{k-1}\lambda_{k-1}(t) + b_{k-1}$, (ii) Nonlinear model if $\lambda_k(t) = \lambda_{k-1}(\alpha_{k-1}t + \beta_{k-1})$, and (iii) Hybrid model if $\lambda_k(t) = a_{k-1}\lambda_{k-1}(\alpha_{k-1}t + \beta_{k-1}) + b_{k-1}$, where $a_k$, $b_k$, $\alpha_k$, and $\beta_k$ are non-negative parameters for $k = 1,2,\ldots$ In this paper, the NHPBP is utilized to derive the stochastic behavior of maintenance or replacement policies through considering the shock-number based aging intensity $\lambda_k(t)$.

Now, we assume a generalized age replacement policy is implemented as follows:

(1) As a NHPBP shock occurs, the system enters one of two types of failure states: type-I failure (minor failure), which is removed by a repair, and type-II failure (catastrophic failure), which is removed by an unplanned (or unscheduled) replacement. A planned (or scheduled) replacement is carried out whenever the system reaches age $T$. Thus, the system is replaced at age $T$ or at any type-II failure, whichever occurs first. A replacement cycle is defined as the time interval between two consecutive replacements.
(2) The probability of a type-II failure depends on the number of shocks suffered since the last replacement. Let $M$ be the number of shocks until the first type-II failure since the last replacement, and $\{\bar{P}_k\}$ be a sequence with the $k$th term $\bar{P}_k = P(M > k)$ as the survival function of $M$ (i.e., $\bar{P}_k$ is the probability that the first $k$ shocks are type-I failures). To model the system deterioration with the number of shocks, we assume that $1 = \bar{P}_0 \geqslant \bar{P}_1 \geqslant \bar{P}_2 \geqslant \cdots$. Let $p_k = P(M = k) = \bar{P}_{k-1} - \bar{P}_k = \bar{P}_{k-1}(1 - \bar{P}_k/\bar{P}_{k-1})$. When the $k$th shock occurs, the system is either repaired (type-I failure) with probability $q_k = \bar{P}_k/\bar{P}_{k-1}$ or replaced (type-II failure) with probability $\theta_k = 1 - q_k = 1 - \bar{P}_k/\bar{P}_{k-1}$.

The following cost structure is imposed on the system:

(1) The costs of unplanned (due to type-II failure) and planned (due to age) replacements are $R_1$ and $R_2$, respectively. We assume that $R_1 \geqslant R_2 > 0$, which signifies that the unplanned replacement cost is no less than planned replacement cost.
(2) A very general random repair cost is considered. The cost of the $k$th general repair at age $t$ is denoted by $g(C(t),c_k(t))$, where $C(t)$ is the age-dependent random part, $c_k(t)$ is the deterministic part which depends on the age and the number of the general repairs, and $g$ is a positive non-decreasing continuous function. The expected repair cost is denoted by $n_k(t) = E_{C(t)}[g(C(t),c_k(t))]$. Assume that the random part $C(t)$ has cumulative distribution function (CDF) $K_t(x)$, probability density function (PDF) $k_t(x)$ and finite first moment $E[C(t)]$. An example of $n_k(t)$ is given in Section 5. Let $m_k(t)$ be the cost per unit time of maintenance of the system at time $t \in [S_k,S_{k+1})$, where $S_k$ is the occurrence time of the $k$th shock for $k = 0,1,2,\ldots$ with $S_0 = 0$.

Moreover, we also make the following assumptions:

(1) The system is monitored continuously and failures are detected immediately.