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Decision Support

Variation analysis of uncertain stationary independent increment processes

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ABSTRACT

A stationary independent increment process is an uncertain process with stationary and independent increments. This paper aims to calculate the variance of stationary independent increment processes, and gains that, for each fixed time, the variance is a constant multiplying the square of time. Based on this result, it is proved that the total variation of stationary independent increment process with finite variance is bounded almost surely. Besides, the quadratic variation of stationary independent increment process with finite variance is 0 almost surely and in mean.

1. Introduction

A stochastic process is a family of random variables from some probability space into a state space. A stochastic process is called Lévy process if it is continuous in probability and has stationary and independent increments. Lévy process is basic in the study of stochastic process admitting jumps. It is well known that Brownian motion, Poisson process, Cauchy process and Γ-process are all examples of Lévy processes. Main original contributions to the theory of Lévy processes are Levy (1934), Khintchine (1937), and Ito (1942) in 1930s and 1940s. The Lévy–Khintchine representation theorem shows that Lévy process can be decomposed to three independent parts (see Sato, 1999). The variance of Lévy process with finite variance has linearity property, namely the variance of Lévy process can be expressed as a constant multiplying time. Besides, the variation of Lévy process is also a fundamental problem. As an example of Lévy process, total variation of Brownian motion is infinite and the quadratic variation is s on the interval $[0, s]$ (see Freedman, 1983; Karatzas and Shreve, 1991; Kallenberg, 1997). However, as another example of Lévy process, the total variation of Poisson is finite and the quadratic is also finite.

In fact, stochastic process is a tool to deal with dynamical random phenomena based on probability theory. Besides probability theory, fuzzy set theory, rough set theory and credibility theory are all introduced to describe non-deterministic phenomena. However, in our daily life, some of the non-deterministic phenomena expressed by the language like “about 100 km”, “approximately 39 °C”, “big size” behave neither like randomness nor like fuzziness. This promotes Liu (2007) to found uncertainty theory, which is a branch of mathematics to deal with human uncertainty based on normality, duality, subadditivity and product axioms. In many cases, the uncertainty is not static, but changes over time. In order to portray dynamic uncertain systems, uncertain process was first introduced by Liu (2008). An uncertain process is essentially a sequence of uncertain variables indexed by time or space. Later, uncertain calculus was proposed by Liu (2009), and the uncertain differential equation driven by canonical process was introduced by Liu (2008). After that, Chen and Liu (2010) proved the existence and uniqueness theorem for uncertain differential equation. Meanwhile, uncertain differential equation has been applied to uncertain optimal control by Zhu (2010), American option pricing by Chen (2011), and other option pricing model by Peng and Yao (2010). Besides, uncertainty theory has been studied by many other researchers Gao (2009), Huang (2011), You (2009), and Zhang (2011).

The concept of stationary independent increment process was proposed by Liu (2008). The canonical process introduced by Liu (2009) and the semi-canonical process introduced by Gao (2011) are examples of stationary independent increment process. Besides, Gao (2011) discussed the properties of the total variation and quadratic variation of semi-canonical process.

In order to discuss the variation of stationary independent increment process, this paper will first discuss the variance of it. Fortunately, we get that the variance of a stationary independent increment process is a constant multiplying the square of time, namely its standard deviation is a constant multiplying the time. Based on this property, we will prove that the total variation of the stationary independent increment process is finite on any bounded interval almost surely, that is, almost all the sample paths of it are of bounded total variation, or almost all the sample paths have finite length on any bounded interval. This property is different from Lévy process, Brown motion as an example of Lévy

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process has infinite length sample paths on any bounded intervals. Besides these, the quadratic variation of stationary independent increment process is also studied, and we also gain that the quadratic variation of stationary independent increment process is 0 almost surely and also in mean. However, the quadratic variation of Brownian motion converges to s on the interval $[0, s]$. Therefore, properties of stationary independent increment process in stochastic process and in uncertain process are totally different.

The rest of the paper is organized as follows. Some preliminary concepts of uncertainty theory are recalled in Section 2. The variance of stationary independent increment process is calculated in Section 3. The total variation of stationary independent increment process is discussed in Section 4. The quadratic variation of stationary independent increment process is studied in Section 5. At last, a brief summary is given in Section 6.

2. Preliminary

Uncertain measure $\mathcal{M}$ is a real-valued set-function on a σ-algebra $\mathcal{L}$ over a nonempty set Γ satisfying normality, duality, subadditivity and product axioms. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 1. Liu (2007). An uncertain variable is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

The uncertainty distribution function $\Phi : \Re \to [0, 1]$ of an uncertain variable ξ is defined as $\Phi(x) = \mathcal{M}\{\xi \leqslant x\}$. The expected value of an uncertain variable is defined as follows:

Definition 2. Liu (2007). Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geqslant r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \leqslant r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

If ξ is an uncertain variable with finite expected value e, then the variance of ξ is defined as $Var[\xi] = E[(\xi - e)^2]$, and the standard deviation is defined as $D[\xi] = \sqrt{Var[\xi]}$. The formula to compute the variance of uncertain variable ξ using its uncertainty distribution Φ is

$$V[\xi] = \int_e^{+\infty} 2(x - e)(1 - \Phi(x))\mathrm{d}x + \int_e^{+\infty} 2(x - e)\Psi(x)\mathrm{d}x \quad (1)$$

where e is the expected value of ξ. Liu (2007) proved the Markov inequality for uncertain variable. Then for any given number $t > 0$ and $p > 0$, we have

$$\mathcal{M}\{|\xi| \geqslant t\} \leqslant \frac{E[|\xi|^p]}{t^p}.$$

Definition 3 Liu (2008). Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set

$$\{X_t \in B\} = \{\gamma \in \Gamma | X_t(\gamma) \in B\}$$

is an event.

From the definition of uncertain process, we know that $|X_t| < +\infty$ almost surely for all $t \geqslant 0$. An uncertain process $X_t(\gamma)$ is a function of time t and γ. For each a fixed time t^*, X_{t^*} is an uncertain variable. For each fixed γ^*, the function $X_t(\gamma)$ is called a sample path of uncertain process X_t. An uncertain process is said to be sample-continuous if almost all sample paths are continuous with respect to t.

Definition 4. An uncertain process X_t is called continuous in measure if it satisfies

$$\lim_{t \to s} \mathcal{M}\{|X_t - X_s| \geqslant \varepsilon\} = 0$$

for any $\varepsilon > 0$.

Definition 5. Liu (2008). An uncertain process X_t is said to have independent increments if

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where t_0 is the initial time and $t_1, t_2, \ldots, t_k$ are any times with $t_0 < t_1 < \cdots < t_k$.

Definition 6. Liu (2008). An uncertain process X_t is said to have stationary increments if, for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertainty variables for all $s > 0$.

Definition 7. Liu (2011). An uncertain process X_t is said to be stationary independent increment process if it has stationary and independent increments.

Liu (2011) proved that the expected value of stationary independent increment process X_t is $E[X_t] = a + bt$ where $a = E[X_0]$ and $b = E[X_1] - a$.

3. Properties of the variance

In this section, we will study the property of the variance of stationary independent increment process. Firstly, we will introduce one theorem that will be used later.

Theorem 1. *Suppose that ξ and η are two independent uncertain variables. Assuming that there exists a nonnegative real number a such that ξ and $a\eta$ have the same uncertainty distribution Φ, then the standard deviation satisfies*

$$D[\xi + \eta] = D[\xi] + D[\eta].$$

Proof. Suppose that $E[\xi] = E[a\eta] = e$, then the expected value of $\xi + \eta$ is $\frac{a+1}{a}e$. Since

$$\mathcal{M}\{\eta \leqslant x\} = \mathcal{M}\{\xi/a \leqslant x\}, \quad (2)$$

we get the uncertainty distribution of $\xi + \eta$ is

$$\Psi(x) = \mathcal{M}\{\xi + \eta \leqslant x\} = \sup_{x_1 + x_2 \leqslant x} \mathcal{M}\{\xi \leqslant x_1\} \wedge \mathcal{M}\{\eta \leqslant x_2\}$$
$$= \sup_{x_1 + x_2 \leqslant x} \Phi(x_1) \wedge \Phi(ax_2) = \Phi(ax/(1 + a)).$$

Using the formula (1), we have

$$Var[\xi + \eta] = \int_{\frac{1+ae}{a}}^{+\infty} 2\left(r - \frac{1 + ae}{a}\right)\left((1 - \Psi(r)) + \Psi\left(\frac{2(1 + a)e}{a} - r\right)\right)\mathrm{d}r$$
$$= \int_{\frac{(1+a)e}{a}}^{+\infty} 2\left(r - \frac{(1 + a)e}{a}\right)\left(1\Phi\left(\frac{ra}{(1 + a)}\right)\right)\mathrm{d}r$$
$$+ \int_{\frac{(1+a)e}{a}}^{+\infty} 2\left(r - \frac{(1 + a)e}{a}\right)\Phi\left(\frac{2(1 + a)e}{a} - r\frac{a}{1 + a}\right)\mathrm{d}r$$