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Innovative Applications of O.R.

Computing arbitrage upper bounds on basket options in the presence of bid–ask spreads

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ABSTRACT

We study the problem of computing the sharpest static-arbitrage upper bound on the price of a European basket option, given the bid–ask prices of vanilla call options in the underlying securities. We show that this semi-infinite problem can be recast as a linear program whose size is linear in the input data size. These developments advance previous related results, and enhance the practical value of static-arbitrage bounds as a pricing technique by taking into account the presence of bid–ask spreads. We illustrate our results by computing upper bounds on the price of a DJX basket option. The MATLAB code used to compute these bounds is available online at www.andrew.cmu.edu/user/jfp/arbitragebounds.html.

1. Introduction

Computing bounds for option prices under incomplete market condition, or under incomplete knowledge of the distribution of the price of the underlying assets is a widely studied pricing technique, where in contrast to parametric pricing techniques, such as Monte Carlo simulations, strong assumptions about the underlying asset price distribution are not required (e.g., consider the recent work of Chen et al. (2011), Primbs (2010), Zuluaga et al. (2008) and Bertsimas and Shah (2008)). These type of bounds provide a mechanism for checking consistency of prices (see, e.g., De la Pena et al., 2004; Hobson et al., 2005a,b), and provide *robust* estimates for option prices in incomplete market conditions, or regardless of any model specifics. Also, these bounds are useful when the number of underlying assets makes the computation of parametric prices numerically challenging. Here, we study the problem of computing arbitrage bounds; that is, computing bounds on the price of an option given the only assumption of absence of arbitrage, and information about prices of other options on the same underlying assets. More specifically, we study the problem of computing the sharpest upper bound on the price of a European basket option, given the only assumption of absence of arbitrage, and information on the bid–ask prices of vanilla European call options on the same underlying assets and with the same maturity. Bounds of this type are called *static-arbitrage bounds.*

The computation of sharp static-arbitrage upper bounds can be formulated as the problem of finding the least expensive portfolio of cash and the options with known prices whose combined payoff super-replicates the payoff of the new basket option of interest (see, e.g., d'Aspremont and El Ghaoui, 2006; Hobson et al., 2005b). This problem has received a fair amount of attention in recent years. Of particular relevance to our work are the recent articles by Albrecher et al. (2008), d'Aspremont and El Ghaoui (2006), Davis and Hobson (2007), Hobson et al. (2005a,b), Laurence and Wang (2005, 2008, 2009) and Peña et al. (2010). In these articles, when formulating the static-arbitrage bound problem, it is assumed that the options can be bought and sold at the same price. In practice the *ask* price, the price at which an investor buys the option, is higher than the *bid* price, the price at which the investor can sell the option. This gives rise to the so-called bid–ask spread. Our approach readily incorporates the use of bid and ask prices in the computation of the super-replicating strategy, thus giving more practical value to the static-arbitrage pricing approach. In particular, this resolves a major limitation in previous approaches (see, e.g., d'Aspremont and El Ghaoui, 2006; Hobson et al., 2005b) that used mid-market prices (e.g., the average of the bid and ask prices) as the "nominal" option prices. Such approximation systematically underestimates the actual buying prices and overestimates the

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actual selling prices. It is then not surprising that the market data used in d'Aspremont and El Ghaoui (2006) and Hobson et al. (2005b) requires a fair amount of "cleaning" to rule out apparent arbitrage opportunities created by these estimates (see Hobson et al., 2005b, Section 6.2).

We undertake a novel approach to the static-arbitrage upper bound problem based entirely on linear programming duality. The foundational block of our work is the construction of an efficient (linear-size) polyhedral description for the set of *super-replicating portfolios*, that is, the set of portfolios of cash and the given options whose payoff super-replicates the basket option's payoff. We show that the set of super-replicating portfolios is a projection of a polyhedron whose description only requires a number of variables and constraints that is linear in the number of given options (see Theorem 1). Although it is intuitively clear that the set of super-replicating portfolios admits a polyhedral description, straightforward attempts to do so yield intractable descriptions that require a number of constraints and variables that is exponentially large in the number of given option prices. It is worth mentioning that these results have been the foundation behind the results developed in Jabbour et al. (2008), Peña et al. (2010) and Peña et al. (2010).

We note that the computation of static-arbitrage *lower* bounds poses a different set of challenges as the nature of *sub-replicating portfolios* is fundamentally different from that of the super-replicating portfolios. The different nature of the upper and lower bound computation has been recognized previously, as it was apparent that the computation of the upper bounds was more tractable (see d'Aspremont and El Ghaoui, 2006; Hobson et al., 2005a,b). In Peña et al. (2010), we present some results for the computation of static-arbitrage lower bounds that are similar in spirit to those discussed herein.

The paper is organized as follows. Section 2 formally presents the problem of computing sharp static-arbitrage upper bounds on a basket option, given the bid–ask prices of vanilla call options on the underlying assets. Also, we present the main building block of our approach; namely, an efficient polyhedral description of the super-replicating portfolios. The latter yields the first efficient linear programming formulation for the computation of static-arbitrage upper bounds that incorporates bid–ask spreads. In Section 3, we provide numerical experiments to illustrate some of our results; namely, we compute bounds of the price of a DJX basket option. The MATLAB code used to compute these bounds is available online at www.andrew.cmu.edu/user/jfp/arbitrage-bounds.html. Finally, Section 4 presents the proofs of the results in the article.

2. Static-arbitrage upper bounds with bid–ask spreads

In this section we present an efficient linear programming formulation for the static-arbitrage bound problem that takes into account bid–ask spreads in the prices of the known options. Previous approaches to the computation of arbitrage bounds (see, e.g., d'Aspremont and El Ghaoui, 2006; Hobson et al., 2005a,b) ignore this important feature and simply assume that the known options can be bought and sold at a mid-market price. This constitutes a major practical limitation as these mid-market prices are rarely arbitrage-free. One of our numerical examples in Section 3 illustrates this phenomenon.

Consider the problem of computing a sharp upper *static-arbitrage* bound on the price of a European basket option, given information on the bid–ask prices of European vanilla options, without making any assumptions other than the absence of arbitrage. This problem can be formulated as the following optimization problem:

$$\begin{aligned}
\inf_{z,y,\bar{y},\underline{y}} \quad & z + \sum_{i=1}^{n}\sum_{j=0}^{m}\bar{p}_{ij}\bar{y}_{ij} - \sum_{i=1}^{n}\sum_{j=0}^{m}\underline{p}_{ij}\underline{y}_{ij} \\
\text{s.t.} \quad & z + \sum_{i=1}^{n}\sum_{j=0}^{m} y_{ij}(s_i - K_{ij})^+ \geqslant \left(\sum_{i=1}^{n}\omega_i s_i - \kappa\right)^+ \quad \text{for all } s \in \mathbb{R}^n_+ \\
& y = \bar{y} - \underline{y} \\
& y \in \mathbb{R}^{n\times(m+1)} \\
& \bar{y}, \underline{y} \in \mathbb{R}^{n\times(m+1)}_+ \\
& z \in \mathbb{R}.
\end{aligned} \tag{1}$$

Above, the multidimensional variable s represents the possible prices of the n underlying assets (at maturity) in the basket. The constants $K_{ij} \in \mathbb{R}$, $i = 1,\dots,n$, $j = 0,\dots,m$, represent the strike price of the call options with payoff $(s_i - K_{ij})^+ := \max\{0, S_i - K_{ij}\}$ whose given ask (buying) and bid (selling) prices are $\bar{p}_{ij} \geqslant \underline{p}_{ij}$ respectively. The vector $\omega \in \mathbb{R}^n$ and constant $\kappa \in \mathbb{R}$ represent the weights and strike of the basket option with payoff $\left(\sum_{i=1}^{n}\omega_i s_i - \kappa\right)^+$ whose price we want to bound. Notice that the assumption on the same number of options m per asset can be made without loss of generality: If one of the assets has fewer than m options, we can artificially increase the number of known options to m by repeating one of the options.

Problem (1) has a natural financial interpretation: It finds the cheapest portfolio of positions in cash (z) and in call options (y_{ij}) with payoff $(s_i - K_{ij})^+$, $i = 1,\dots,n$, $j = 0,\dots,m$ that super-replicates the payoff of the basket option with payoff $\left(\sum_{i=1}^{n}\omega_i s_i - \kappa\right)^+$.

Following d'Aspremont and El Ghaoui (2006), we implicitly assume that all the options have the same maturity, and that the risk-free interest rate is zero; or equivalently, we compare the prices in the forward market (at maturity).

2.1. Super-replication of a linear payoff

Now we present the main building block of our approach; namely, an efficient polyhedral description of the super-replicating constraint (first constraint) in problem (1). The latter yields the first efficient linear programming formulation for the computation of static-arbitrage upper bounds that incorporates bid–ask spreads.

For ease of notation, let us first rewrite problem (1) in "vector form". That is,

$$\begin{aligned}
\inf_{z,y,\bar{y},\underline{y}} \quad & z + \sum_{j=0}^{m}\bar{p}^j\cdot\bar{y}^j - \sum_{j=0}^{m}\underline{p}^j\cdot\underline{y}^j \\
\text{s.t.} \quad & z + \sum_{j=0}^{m} y^j\cdot(s - K^j)^+ \geqslant (\omega\cdot s - \kappa)^+ \quad \text{for all } s \in \mathbb{R}^n_+ \\
& y = \bar{y} - \underline{y} \\
& y \in \mathbb{R}^{n\times(m+1)} \\
& \bar{y}, \underline{y} \in \mathbb{R}^{n\times(m+1)}_+ \\
& z \in \mathbb{R},
\end{aligned} \tag{2}$$

where a^j denotes the vector $[a_{ij}]_{i=1,\dots,n}$, and $(\cdot)$ denotes the dot (inner) product of vectors.

Now, assume $K = [K^0 \quad K^1 \quad \cdots \quad K^m] \in \mathbb{R}^{n\times(m+1)}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ are given. Define the set of *super-replicating strategies* $SR(K, b, c)$ as follows

$$SR(K,b,c) := \left\{(y,z) = (y^0, y^1, \dots, y^m, z) \in \mathbb{R}^{n\times(m+1)} \times \mathbb{R} : z + \sum_{j=0}^{m} y^j\cdot(s - K^j)^+ \geqslant b\cdot s - c \text{ for all } s \in \mathbb{R}^n_+\right\}. \tag{3}$$