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Avoiding unfairness of *Owen* allocations in linear production processes

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ABSTRACT

This paper deals with cooperation situations in linear production problems in which a set of goods are to be produced from a set of resources so that a certain benefit function is maximized, assuming that resources not used in the production plan have no value by themselves. The *Owen* set is a well-known solution rule for the class of linear production processes. Despite their stability properties, *Owen* allocations might give null payoff to players that are necessary for optimal production plans. This paper shows that, in general, the aforementioned drawback cannot be avoided allowing only allocations within the core of the cooperative game associated to the original linear production process, and therefore a new solution set named *EOwen* is introduced. For any player whose resources are needed in at least one optimal production plan, the *EOwen* set contains at least one allocation that assigns a strictly positive payoff to such player.

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1. Introduction

A benefit cooperative game is a pair (N, v), where $N = \{1, 2, \dots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function assigning to every coalition $S \subset N$ the maximum benefit that the cooperation between players in S would yield. For a complete introduction on cooperative game theory see for instance Owen (1995) or Forgó et al. (1999). Assuming that the game is superadditive, that is $v(S) + v(T) \leqslant v(S \cup T)$, $\forall S, T \subset N$, cooperation among all players is beneficial and, therefore, the *grand coalition N* is to form.

One of the main questions in cooperative game theory is how to distribute the benefit obtained by the grand coalition N among the players. An allocation is a vector $\alpha \in \mathbb{R}^n$, such that α_i is the payoff of player i and $\sum_{i=1}^n \alpha_i = v(N)$. One well-accepted way of allocating $v(N)$ among the players is to find allocations in the *core*. The core of a game (N, v), denoted by $Core(N, v)$, is the set of allocations satisfying that no coalition of players can obtain a better payoff by acting separately from the rest of players. That is,

$$Core(N, v) = \{\alpha \in \mathbb{R}^n : v(S) \leqslant \alpha(S) \ \forall S \subset N, \ \ v(N) = \alpha(N)\}, \qquad (1)$$

where $\alpha(S) = \sum_{i \in S} \alpha_i, \ \forall S \subset N$. In principle, the core has at least two problems: the core of a game might be empty, that is, there are games for which no core allocations exist, and finding a core allocation might be a NP-hard problem. Along the years, many other allocation rules have appeared in the literature. One of the most used allocation rules is the Shapley value, which has attracted a lot of interest for its many applications, see Moretti and Patrone (2008).

A linear production problem is a situation in which certain goods that can be sold in a market are to be produced from a set of available distinct resources. An implicit feature of the linear production problems we deal with in this paper is that the resources not used in the production plan have no value at all. Situations like this may arise when the resources are perishable and, if not used in the next production plan, they are wasted. Another example of this situation is found in some industries in developed countries that give their excesses to underdeveloped countries, to charity organizations, or even to other companies within the same area as long as they are not competing ones. This is beneficial for both parties: the donor party gets rid of excesses which, if not used, must be eliminated at certain cost, and the receiving party only has to pay for the shipping costs, which is usually cheaper than having to buy the material.

In this paper we study a new set of allocations for *linear production processes* (LP processes for short), which arise when a bunch of players $N = \{1, \dots, n\}$ with conflicting objectives control the resources of a LP problem. A cooperative game, called LP game, can be associated to each LP process. (Note that different LP processes may generate the same LP game.) An early reference to LP games can be found in Owen (1975). LP games are totally balanced games, so every subgame of a LP game has a non-empty core. By solving the dual problem of the underlying linear production problem we can obtain a set of allocations for LP processes known as *Owen* allocations (see Owen (1975)), which has been well-studied in the literature. One of its main properties is that *Owen* allocations are always core allocations, and are easily computed. More recently,

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Van Gellekom et al. (2000) provided an axiomatic characterization of this solution set. In this paper we show that, despite their stability properties, *Owen* allocations do not always yield a *fair* distribution of the benefit obtained. For instance, a player whose resources are necessary for any optimal plan may receive a null payoff from *Owen* allocations. Such drawback is discussed in this paper, and an alternative allocation set is proposed.

Since the pioneering work by Owen, several generalizations of LP games have appeared in the literature. Dubey and Shapley (1984) study a game in which players have partial control over the constraints of a general mathematical programming problem. Granot (1986) introduces another generalization in which the resources owned by a coalition are not restricted to be the sum of the resources of players in the coalition. Curiel et al. (1989) introduce LP games with committee control, obtaining results on the balancedness of these games, whose core has been more recently studied by Molina and Tejada (2004).

The goal of this paper is to introduce a new set of allocations for linear production processes that avoid some of the aforementioned drawbacks of the *Owen* set. To this end, the rest of the paper is structured as follows. Section 2 gives a short introduction to LP processes and a motivation of the studied problem. Some definitions and technical results are given in Section 3. The allocation set proposed in this paper is introduced and analyzed in Section 4. An axiomatic characterization and some of its properties are given, as well as a discussion about the impossibility of finding core allocations that avoid the unfairness problem of the *Owen* allocations we address in this paper.

2. Linear production processes

A LP problem is a situation in which there is a finite set of resources $R = \{1,2,\ldots,r\}$ and from those resources a set $P = \{1,2,\ldots,p\}$ of consumption goods can be produced. The production technologies are given by a matrix $A \in \mathbb{R}^{r\times p}$, where $A_{ij} \geqslant 0$ denotes the amount of resource i necessary to produce one unit of product j, $\forall i = 1,\ldots,r$, $j = 1,\ldots,p$. It is also assumed that the demand of every product is large enough to sell all produced products, the unitary market price of product j being $c_j \geqslant 0$. The objective of a LP problem is to decide how much of each product should be produced so that the general benefit is maximized.

Assume now that a group of players $N = \{1,\ldots,n\}$ control the resources $R = \{1,2,\ldots,r\}$, that is, player k owns $B_{ik} \geqslant 0$ units of resource $i, k = 1,\ldots,n$, $i = 1,\ldots,r$. Therefore, let $B = (B_{ik})_{r\times n}$ be the resource-player matrix. Let $b \in \mathbb{R}^r$ be the resource vector, that is $b = Be_N$, where $e_S \in \mathbb{R}^n$ satisfying $(e_S)_k = 1$ if $k \in S$, and zero otherwise for all $S \subseteq N$. In other words, b_i is the total amount of resource i owned by the grand coalition, that is, $b_i = \sum_{k=1}^{n} B_{ik}$ $\forall i \in R$. Thus, the maximum profit that can be made by the cooperation of all players is the value of problem P_N:

$$\begin{array}{llll} \max & cx & \min & yb \\ \text{s.t.} & Ax \leqslant b \quad (P_N), & \text{s.t.} & yA \geqslant c \quad (D_N), \\ & x \geqslant 0 & & y \geqslant 0 \end{array} \tag{2}$$

where D_N is the dual problem of P_N (see Bazaraa et al. (1990) for a description of duality theory in linear programming). It is easy to check that, although players can try to produce separately, it is always more profitable to join their resources since the benefit they obtain this way is at least as high as the sum of the possible coalitions' profits separately. For a coalition $S \subset N$, we define its characteristic function, $\upsilon(S)$, via the optimal value of problem P_S:

$$\begin{array}{llll} \max & cx & \min & yBe_S \\ \text{s.t.} & Ax \leqslant Be_S \quad (P_S), & \text{s.t.} & yA \geqslant c \quad (D_S), \\ & x \geqslant 0 & & y \geqslant 0 \end{array} \tag{3}$$

where D_S is the dual of P_S.

Problem P_S is feasible and bounded for all possible coalitions if $Be_S > 0$, $c \geqslant 0$ and $\forall j$: $c_j > 0$ there is at least one resource $i \in R$ with $A_{ij} > 0$.

Each triple (A,B,c) satisfying the conditions above will be called in the following, according to Van Gellekom et al. (2000), a *linear production process*. Let $\mathcal{L}$ denote the class of LP processes. From the definition of the characteristic function υ one can associate to each LP process a cooperative game (N,υ). The reader may note that the same LP game can originate from different LP processes.

Now a natural question arises: how to divide the profit made by the grand coalition among the players. Let us introduce some notation that will be useful in the rest of the paper.

Let $(A,B,c) \in \mathcal{L}$. The feasible regions of problems P_N and D_N, see (2), are denoted by

$$\begin{aligned} F_{\max}(A,B,c) &:= \{x \in \mathbb{R}^p_+ : Ax \leqslant b\}, \\ F_{\min}(A,B,c) &:= \{y \in \mathbb{R}^n_+ : yA \geqslant c\}, \end{aligned} \tag{4}$$

respectively. The optimal values of problems P_N and D_N are denoted by

$$\begin{aligned} \upsilon_{\max}(A,B,c) &:= \max\{cx : x \in F_{\max}(A,B,c)\}, \\ \upsilon_{\min}(A,B,c) &:= \min\{yb : y \in F_{\min}(A,B,c)\}, \end{aligned} \tag{5}$$

respectively, and the set of optimal solutions to P_N and D_N by

$$\begin{aligned} O_{\max}(A,B,c) &:= \{x \in F_{\max}(A,B,c) : cx = \upsilon_{\max}(A,B,c)\}, \\ O_{\min}(A,B,c) &:= \{y \in F_{\min}(A,B,c) : yb = \upsilon_{\min}(A,B,c)\}. \end{aligned} \tag{6}$$

A *solution rule* φ on $\mathcal{L}$ is a map assigning to every LP process $(A,B,c) \in \mathcal{L}$ a set $\Gamma \subset \mathbb{R}^n$ such that $\sum_{i\in N}\gamma_i = \upsilon_{\max}(A,B,c)$ for all $\gamma \in \Gamma$. Each member of this set is an *allocation*. A well-known solution rule for cooperative games is the core, see (1). One well-accepted solution rule specific for LP processes is the *Owen set*, defined from optimal solutions to the dual problem D_N.

Definition 1. Let $(A,B,c) \in \mathcal{L}$. The *Owen* set of (A,B,c) is

$$Owen(A,B,c) := \{yB : y \in O_{\min}(A,B,c)\}. \tag{7}$$

Owen (1975) proved that $Owen(A,B,c) \subseteq Core(A,B,c)$ for every $(A,B,c) \in \mathcal{L}$. That is, *Owen* allocations are stable in the sense that no group of players can obtain a better payoff by acting separately. Despite these good properties, they should not be considered as ideal allocations. See the following example.

Example 1. Consider the 3-player game $(A,B,c) \in \mathcal{L}$ where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The corresponding dual problem $D(N)$ is

$$\begin{array}{ll} \min & 2y_1 + 4y_2 + y_3 + 5y_4 \\ \text{s.t.} & y_1 + y_2 + y_4 \geqslant 1, \\ & y_2 + y_3 + 2y_4 \geqslant 2, \\ & y_1, y_2, y_3, y_4 \geqslant 0. \end{array} \tag{8}$$

The characteristic function of the associated game is $\upsilon(\{i\}) = \upsilon(\{1,3\}) = 0$ $\forall i = 1,2,3$, $\upsilon(\{1,2\}) = 3$, $\upsilon(\{2,3\}) = 1$, $\upsilon(\{1,2,3\}) = 4$. It can be checked that $O_{\min}(A,B,c) = \{(1,0,2,0)\}$ and, therefore, $Owen(A,B,c) = \{(1,0,2,0)B\} = \{(3,0,1)\}$.

This allocation is in the core of the game but, is it a "fair" allocation? Note that player 2 receives nothing but, without his resources, the optimal production plan cannot be achieved. So, the

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