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Invited Review

Bin packing and cutting stock problems: Mathematical models and exact algorithms

Maxence Delorme[a], Manuel Iori[b], Silvano Martello[a,*]

[a] DEI "Guglielmo Marconi", Alma Mater Studiorum - Università di Bologna, Viale Risorgimento 2, 40136 Bologna, Italy
[b] DISMI, Università di Modena e Reggio Emilia, Via Giovanni Amendola 2, 42122 Reggio Emilia, Italy

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ABSTRACT

We review the most important mathematical models and algorithms developed for the exact solution of the one-dimensional bin packing and cutting stock problems, and experimentally evaluate, on state-of-the art computers, the performance of the main available software tools.

1. Introduction

The (one-dimensional) bin packing problem is one of the most famous problems in combinatorial optimization. Its structure and its applications have been studied since the thirties, see Kantorovich (1960). Gilmore and Gomory (1961) introduced, for this class of problems, the concept of column generation, by deriving it from earlier ideas of Ford and Fulkerson (1958) and Dantzig and Wolfe (1960). This is one of the first problems for which, since the early seventies, the worst-case performance of approximation algorithms was investigated. In the next decades lower bounds were studied and exact algorithms proposed. As the problem is strongly $\mathcal{NP}$-hard, many heuristic and metaheuristic approaches have also been proposed along the years.

The *bin packing problem* (BPP) can be informally defined in a very simple way. We are given n *items*, each having an integer *weight* w_j ($j = 1, \ldots, n$), and an unlimited number of identical *bins* of integer *capacity* c. The objective is to pack all the items into the minimum number of bins so that the total weight packed in any bin does not exceed the capacity. (In a different but equivalent *normalized* definition, the weights are real numbers in [0, 1], and the capacity is 1.) We assume, with no loss of generality, that $0 < w_j < c$ for all j.

Many variants and generalizations of the BPP arise in practical contexts. One of the most important applications, studied since the sixties, is the *Cutting Stock Problem* (CSP). Although it has been defined in different ways according to specific real world cases, its basic definition, using the BPP terminology, is as follows. We are given m *item types*, each having an integer *weight* w_j and an integer *demand* d_j ($j = 1, \ldots, m$), and a sufficiently large number of identical *bins* of integer *capacity* c. (In the CSP literature the bins are frequently called *rolls*, the term coming from early applications in the paper industry, and "cutting" is normally used instead of "packing".) The objective is to produce d_j copies of each item type j (i.e., to cut/pack them) using the minimum number of bins so that the total weight in any bin does not exceed the capacity.

This paper is devoted to a presentation of the main mathematical models that have been proposed, and to an experimental evaluation of the main available software tools that have been developed. The main motivations for writing this survey are to present, for the first time, a complete overview on these problems and to assess, through extensive computational experiments, the performance of the main computer codes that are available for their optimal solution. All the codes we evaluated are either linked or downloadable from a dedicated web page, but one that can be obtained by the authors. The same web page also provides the test instances we used, including new instances that were specifically created as challenging test cases. We believe that this study and the accompanying web page will be useful to many researchers who are still intensively studying this area. Indeed, a search on different bibliographic data bases for articles having in the title either the term "bin packing", or the term "cutting stock", or both, shows a growing interest in these problems in the last 25 years, with sharp increase in recent years (over 150 Google Scholar entries in 2015).

* Corresponding author. Tel.: +39 051 2093022; fax: +39 051 2093073.
E-mail address: silvano.martello@unibo.it, silvano.martello@gmail.com (S. Martello).

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For exhaustive studies on specific research areas concerning the BPP and the CSP, the reader is referred to many surveys that have been published along the years. To the best of our knowledge, the following reviews have been proposed.

The first literature review on these problems was published by Sweeney and Paternoster (1992), who collected more than 400 books, articles, dissertations, and working papers appeared from 1961 to 1990. Dyckhoff (1990) proposed a typology of cutting and packing problems, and classified the BPP and the CSP as 1/V/I/M and 1/V/I/R, respectively. In the same year Martello and Toth included a chapter on the BPP in their book (Martello & Toth, 1990a) on knapsack problems. Two years later Dyckhoff and Finke (1992) published a book on cutting and packing problems arising in production and distribution, where they investigated the different structure of these problems, and classified the literature accordingly. A bibliography on the BPP has been compiled by Coffman, Csirik, Johnson, and Woeginger (2004). More recently, Wäscher, Haußner, and Schumann (2007) re-visited the typology by Dyckhoff (1990) and proposed more detailed categorization criteria: the problems we consider are classified as 1-dimensional SBSBPP (*Single Bin Size Bin Packing Problem*) and 1-dimensional SSSCSP (*Single Stock Size Cutting Stock Problem*).

Besides the general surveys discussed above, a number of reviews concerning specific methodologies have been proposed. Already in the early eighties Garey and Johnson (1981) and Coffman, Garey, and Johnson (1984) presented surveys on approximation algorithms for the BPP. Other surveys on approximation algorithms for the BPP and a number of its variants were later proposed by Coffman, Galambos, Martello, and Vigo (1999), Coffman, Garey, and Johnson (1996) and Coffman and Csirik (2007b). Coffman and Csirik (2007a) also proposed a four-field classification scheme for papers on bin packing, aimed at highlighting the results in bin packing theory to be found in a certain article. More recently, Coffman, Csirik, Galambos, Martello, and Vigo (2013) presented an overview of approximation algorithms for the BPP and a number of its variants, and classified all references according to Coffman and Csirik (2007a).

Valério de Carvalho (2002) presented a survey of the most popular *Linear Programming* (LP) methods for the BPP and the CSP. A review of models and solution methods was included by Belov (2003) in his PhD thesis dedicated to one- and two-dimensional cutting stock problems.

We finally mention that extensions to higher dimensions have been investigated too. In the early nineties, Haessler and Sweeney (1991) provided a description of one- and two-dimensional cutting stock problems, and a review of some of the methods to solve them. More recently, surveys on two-dimensional packing problems have been presented by Lodi, Martello, and Monaci (2002), Lodi, Martello, Monaci, and Vigo (2010) and Lodi, Martello, and Vigo (2002).

In the next section we provide a formal definition of the BPP and the CSP. In Section 3 we briefly review the most successful upper and lower bounding techniques for the considered problems. In Sections 4–6 we examine pseudo-polynomial formulations, enumeration algorithms, and branch-and-price approaches, respectively. Finally, in Section 7, we experimentally evaluate the computational performance of twelve computer programs available for the solution of the considered problems. Conclusions follow in Section 8.

2. Formal statement

In order to give a formal definition of the problems, let u be any upper bound on the minimum number of bins needed (for example, the value of any approximate solution), and assume that the potential bins are numbered as $1, \dots, u$. By introducing two types of binary decision variables

$$y_i = \begin{cases} 1 & \text{if bin } i \text{ is used in the solution;} \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, \dots, u),$$

$$x_{ij} = \begin{cases} 1 & \text{if item } j \text{ is packed into bin } i; \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, \dots, u;\ j = 1, \dots, n),$$

we can model the BPP as a basic *Integer Linear Program* (ILP) of the form (see Martello & Toth, 1990a)

$$\min \sum_{i=1}^{u} y_i \tag{1}$$

$$\text{s.t.} \sum_{j=1}^{n} w_j x_{ij} \le c y_i \quad (i = 1, \dots, u), \tag{2}$$

$$\sum_{i=1}^{u} x_{ij} = 1 \quad (j = 1, \dots, n), \tag{3}$$

$$y_i \in \{0, 1\} \quad (i = 1, \dots, u), \tag{4}$$

$$x_{ij} \in \{0, 1\} \quad (i = 1, \dots, u;\ j = 1, \dots, n). \tag{5}$$

Constraints (2) impose that the capacity of any used bin is not exceeded, while constraints (3) ensure that each item is packed into exactly one bin.

For the CSP let us define u and y_i as above, and let

$$\xi_{ij} = \text{number of items of type } j \text{ packed into bin } i \quad (i = 1, \dots, u;\ j = 1, \dots, m).$$

The CSP is then

$$\min \sum_{i=1}^{u} y_i \tag{6}$$

$$\text{s.t.} \sum_{j=1}^{m} w_j \xi_{ij} \le c y_i \quad (i = 1, \dots, u), \tag{7}$$

$$\sum_{i=1}^{u} \xi_{ij} = d_j \quad (j = 1, \dots, m), \tag{8}$$

$$y_i \in \{0, 1\} \quad (i = 1, \dots, u), \tag{9}$$

$$\xi_{ij} \ge 0, \text{integer} \quad (i = 1, \dots, u;\ j = 1, \dots, m). \tag{10}$$

The BPP can be seen as a special case of the CSP in which $d_j = 1$ for all j. In turn, the CSP can be modeled by a BPP in which the item set includes d_j copies of each item type j.

The BPP (and hence the CSP) has been proved to be $\mathcal{NP}$-hard in the strong sense by Garey and Johnson (1979) through transformation from the 3-Partition problem.

3. Upper and lower bounds

Most exact algorithms for bin packing problems make use of upper and lower bound computations in order to guide the search in the solution space, and to fathom partial solutions that cannot lead to optimal ones. As previously mentioned, for deep reviews on these specific domains, the reader is referred to the surveys listed in Section 1. In this section we briefly review the most successful upper and lower bounding techniques that have been developed, with some focus on areas for which no specific survey is available. We use the term *approximation algorithm* for methods for which theoretical results (like, e.g., worst-case performance) can be established, while the term *heuristic* denotes methods for which the main interest relies in their practical behavior.

A classical way for evaluating upper and lower bounds is their absolute worst-case performance ratio. Given a minimization problem and an approximation algorithm A, let $A(I)$ and $OPT(I)$ be the