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Discrete Optimization

The constrained shortest path problem with stochastic correlated link travel times

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ABSTRACT

This paper investigates the constrained shortest path problem in a transportation network where the link travel times are assumed to be random variables defined on the basis of joint probability mass functions. A 0–1 integer programming model is formulated to find the least expected travel time paths. Besides the flow balance and side constraints, the unique link selection constraint is particularly introduced to guarantee that only an optimal path can be finally generated. Then, a Lagrangian relaxation solution approach is provided to relax the hard constraints and decompose the relaxed model into two parts. An algorithmic framework, integrating the sub-gradient algorithm, label-correcting algorithm and *K*-shortest path algorithm, is designed to minimize the gap between the upper and lower bounds to find near-optimal solutions. An extension that considers the joint probability mass functions of link travel times varying with time is discussed. We decompose the time-dependent problem into three subproblems and solve it by the modified algorithmic framework. Computational tests are conducted on different scales of transportation networks. Experimental results in a large-scale network indicate that the proposed algorithm can quickly find the high-quality solutions with small relative gaps, demonstrating the effectiveness of the proposed approaches.

1. Introduction

The problem of selecting a path with the minimum travel time in a transportation network is termed as standard shortest path (SP) problem, which can be solved optimally via some efficient algorithms (Dantzig, 1960; Dijkstra, 1959; Floyd, 1962). However, if one desires to select a path not only with the minimum travel time but also with other resources (e.g., length, cost) that can not exceed predetermined values, this problem will not be a standard SP problem. Therefore, one modified version of the standard SP problem, i.e., constrained shortest path (CSP) problem, is first studied by Witzgall and Goldman (1965) and Joksch (1966), which includes one or more side (additional) constraints that establish upper limits on the sums of some other arc weights for the path. This problem has numerous applications in the real world: the crew-pairing, crew-scheduling and crew-rostering problems in the airline industry (Gamache, Soumis, Marquis, & Desrosiers, 1999; Lavoie, Minoux, & Odier, 1988; Salazar-González, 2014); minimum-risk routing of vehicles and aircraft (Latourell, Wallet, & Copeland, 1998; Lee, 1995); time-dependent shortest path problems (Nachtigall, 1995; Adil, Turkay, & Ibrahim, 2006); multicriteria (multiobjective) shortest path problem (Martins, 1984; Tarapata, 2007), by minimizing one of the objectives and making the others as side constraints; civil engineering problem (Elimam & Kohler, 1997), by designing wastewater treatment model and efficient composite structures; vehicle routing problems (Rivera, Afsar, & Prins, 2016; Tas, Gendreau, Dellaert, Van Woensel, & De Kok, 2014), which can be handled based on the constrained shortest path formulation; and numerous network optimization problems (Yang, Li, Gao, & Li, 2012; Yang, Zhou, & Gao, 2014).

In general, unlike SP problem, the CSP problem can not be solved directly by the label-setting or label-correcting algorithms, since the additional constraints cause the CSP problem to be an NP-hard problem (Michael & David, 1979). Up to now, researchers have proposed a variety of efficient algorithms to solve the CSP problem in recent decades. (1) Dynamic programming. Witzgall and Goldman (1965) used dynamic programming to solve the CSP problem with one side constraint. Joksch (1966) extended the dynamic programming approach to solve the CSP problem with multiple side constraints. (2) Label-setting algorithm based on dynamic programming (Aneja, Aggarwal, & Nair, 1983). This algorithm can improve computational efficiency of dynamic programming. (3) Lagrangian relaxation combined with *K*-shortest path algorithm (Handler & Zang, 1980). (4) Branch-and-Bound approach based on Lagrangian relaxation (Beasley & Christofides, 1989).

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Recently, researchers are particularly interested in more efficient algorithms for solving large-scale CSP problems. For instance, a new heuristic, i.e., penalty function heuristic (PFH), was proposed by Avella, Boccia, and Sforza (2002) for solving the large-scale CSP problem effectively. In Boland, Detheridge, and Dumitrescu (2006), a state-space augmenting approach was applied to the label setting algorithm for accelerating computational times. Santos, Coutinho-Rodrigues, and Current (2007) introduced a new optimal algorithm through defining a more efficient search direction based on the K-shortest path algorithm, which is superior to the previous ones in computational time. Zhu and Wilhelm (2007, 2012) presented a three-stage approach to solve the constrained shortest path problem which was used as a subproblem in the column-generation context. Pugliese and Guerriero (2013) addressed elementary shortest path problem with forbidden paths. This problem could be formulated as a specific instance of the resource constrained shortest path problem and two different solution approaches were defined and implemented. Lozano and Medaglia (2013) presented an exact solution method (pulse algorithm) to handle large-scale networks in a reasonable time. Zhang, Zhang, Hu, Deng, and Mahadevan (2013) proposed an adaptive amoeba model and designed an amoeba algorithm combined with the Lagrangian relaxation algorithm to solve the CSP problem effectively. Ma (2014) proposed a fast A*-based label setting algorithm to solve the multimodal resource CSP problem efficiently, in which a speed-up technique based on the A*-algorithm and Access-Node routing was designed to reduce the search space. Bode and Irnich (2014) used a new labeling algorithm to handle the combined (k, 2)-loop elimination for solving the shortest path problem with resource constraints.

Note that, in actual traffic conditions, transportation networks are inherently uncertain. Random events such as incidents, vehicle breakdown, bad weather, road construction and special activities greatly affect the reliability of transportation networks. In order to model travelers' route choice decisions, a stochastic network is required to capture such uncertainties, where link travel times are random variables. This problem is usually defined as the stochastic shortest path (SSP) problem. For example, Frank (1969) addressed the problem of determining reliability-based optimal path in stochastic networks by using the probability distributions of link travel times. Bajia (1985) determined the probability distribution function of duration of the shortest path in a stochastic network. The minimum expected disutility theory based on the von Neumann and Morgenstern risk-decision paradigm (von Neumann & Morgenstern, 1944) was also a common evaluation criterion for optimal path. Eiger, Mirchandani, and Soroush (1985) found optimal paths with disutility functions by the Bellman's principle of optimality when affine or exponential functions were used. Murthy and Sarkar (1998) determined optimal paths by minimizing the piecewise-linear and concave utility functions. Thus nonlinear disutility functions can be approximated by such disutility functions to capture risk averse behavior. The mean-variance tradeoff was another way for handling the stochastic variables. Sivakumar and Batta (1994) defined the minimum expected travel time path with a variance smaller than a benchmark by adding a constraint into the problem. Sen, Pillai, Joshi, and Rathi (2001) made the objective function become a parametric linear combination of mean and variance. Yang, Yang, and You (2013); Yang, Zhang, Li, and Gao (2016) formulated two-stage stochastic optimization models by using the random scenario-based link travel times. Additionally, some researchers also considered the time-dependent characteristics of link travel times to capture the dynamic nature of transportation networks (e.g., Gao & Chabini, 2006; Huang & Gao, 2012; Miller-Hooks, 2001; Nielsen, Kim Allan Andersen, & Pretolani, 2014; Wang, Gao, & Yang, 2015; 2016).

To the best of our knowledge, the CSP problem is only considered in deterministic conditions up to now, while the CSP problem in stochastic conditions has not yet been concerned in the literature. In light of that, it is necessary and significant to consider the CSP problem in stochastic conditions according to the above discussions for uncertain transportation networks. In this paper, we aim to find the least expected travel time path satisfying one or more side constraints in a stochastic transportation network, where the correlation of link travel times is represented by the joint probability distribution functions. For convenience, the problem is hereafter referred to as the stochastic constrained shortest path (SCSP). We intend to accomplish the following research contributions.

(1) This paper first proposes the SCSP problem, in which the random link travel times are used to show the uncertainty of transportation networks. Meanwhile, in order to capture the correlation of link travel times, this study uses the joint probability mass functions over the considered network to represent the random link travel times. The SSP problem has been well studied by numerous researchers, yet no CSP problems are discussed in stochastic environment up to now. This is essentially because the CSP problem itself is an NP-hard problem, and the computational complexity becomes much larger if the link travel times are assumed to be uncertain.

(2) For the SCSP problem, a 0–1 integer programming model is formulated to find the a priori optimal path with the least expected travel time, in which the unique link selection constraint is introduced to ensure that only an optimal path is generated finally. This constraint is reformulated as its equivalent form to reduce the redundancy and the range of Lagrangian multipliers' value.

(3) The third emphasis of this paper is placed on designing a Lagrangian relaxation-based algorithmic framework to solve the proposed SCSP model. The Lagrangian relaxation approach is developed to relax hard constraints, i.e., side constraints and unique link selection constraints, into the objective function. The relaxed model can be further decomposed into two parts. One is a standard SP problem over each support point, which can be solved by the label correcting algorithm; the other is considered as a constant, which can be obtained by the given Lagrangian multipliers and parameters. The sub-gradient algorithm is offered to minimize the gap between the upper and lower bounds (see Section 3.2 for details), in which the K-shortest path algorithm is embedded to deal with the infeasibility of generated paths in the process of solving relaxed model.

(4) Finally, an extension model considering time-dependent characteristic of link travel times is further discussed. For solving this time-dependent problem which is related to arrival times at intermediate nodes, a modified label-correcting algorithm is embedded into the algorithmic framework. Numerical experiments demonstrate the effectiveness of the Lagrangian relaxation-based algorithmic framework for solving the time-dependent model. It is worthwhile to note that the time-dependent case can reflect more superiority of the proposed algorithm.

The remainder of this paper is organized as follows. Section 2 presents the SCSP problem and formulates it as an integer programming model. The solution methodology for the proposed model is designed in Section 3. Section 4 extends the SCSP model to its time-dependent case. Numerical experiments are implemented in Section 5 to test the performance and effectiveness of the proposed model and solution algorithm. Section 6 concludes the paper and proposes the future research directions.

2. Problem statements

Let $G=(N,A,T)$ be a directed, well-connected network. $N(|N|=n)$ is the set of nodes and $A(|A|=m)$ is the set of links. Each link $(i,j)\in A$ connects two distinct nodes $i,j\in N$. A directed