European Journal of Operational Research 255 (2016) 103–109

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Stochastics and Statistics

Probability weighting and L-moments[☆]

Pavlo Blavatskyy[a,b,*]

[a] School of Business and Governance, Murdoch University, 90 South Street, Murdoch, WA 6150, Australia
[b] Montpellier Business School, Montpellier Research in Management, 2300 Avenue des Moulins, 34080 Montpellier, France

ARTICLE INFO

Article history:
Received 7 December 2015
Accepted 6 May 2016
Available online 31 May 2016

Keywords:
Decision under risk
Probability weighting function
Cumulative prospect theory
Mean-Gini approach
Theory of L-moments

ABSTRACT

Several popular generalizations of expected utility theory—cumulative prospect theory, rank-dependent utility and Yaari's dual model—allow for non-linear transformation of (de-)cumulative probabilities. This paper shows an unexpected connection between probability weighting and the statistical theory of L-moments. Specifically, cubic probability weighting results in a linear tradeoff between the expected value (the first L-moment), Gini (1912) mean difference statistic (the second L-moment, also known as L-scale) and the third L-moment (measuring skewness). Inverse S-shaped probability weighting function crossing the 45° line at a probability ≤0.5 reflects an aversion to the dispersion of outcomes and an attraction to positively skewed distributions.

1. Probability weighting and L-moments

The Allais (1953) paradox highlighted descriptive limitations of expected utility theory—people may reveal a different preference ordering over two pairs of probability distributions that must be ranked consistently by any expected utility maximizer. In response to the Allais (1953) paradox, expected utility theory was generalized to numerous non-expected utility theories (reviewed in Starmer, 2000). Popular generalizations of expected utility theory that can rationalize several behavioral regularities in choice under risk/uncertainty are Tversky and Kahneman (1992) cumulative prospect theory, [1] Quiggin (1981) rank-dependent utility and Yaari (1987) dual model. These theories introduce a non-linear probability weighting function over (de-)cumulative probabilities in choice under risk (or non-additive capacities over events in choice under uncertainty/ambiguity).

This paper shows an unexpected connection between probability weighting and the statistical theory of L-moments. Under Yaari (1987) dual model with a cubic probability weighting function preferences are represented by a weighted sum of three statistical measures: (1) the expected value of a lottery (which is also the first L-moment); (2) Gini (1912) mean difference statistic[2] (or the second L-moment, which is sometimes called L-scale); and (3) the third L-moment of a lottery (a measure of skewness). Thus, there is an unexpected connection to the financial literature.

Markowitz (1952) assumed that investor's preferences depend not only on the expected value (the mean) but also on the standard deviation (or the variance) of assets' returns. Unfortunately, any investor with such preferences inevitably violates the first-order stochastic dominance (*cf.* Borch 1969). Yitzhaki (1982) showed that violations of stochastic dominance can be avoided by using a different measure of statistical dispersion of assets' returns—Gini (1912) mean difference statistic.[3]

The mean-Gini approach of Shalit and Yitzhaki (1984) can be further extended by introducing a preference for gambling. Already Markowitz (1952, p. 90) considered the possibility that investors may care not only about the mean and the standard deviation (or the variance) but also—about the skewness of assets' returns. Yet, Markowitz (1952, p. 90) proposed to measure skewness with the third central moment, which may lead to the violations of the first-order stochastic dominance. Such violations may be avoided by

☆ I am grateful to the participants of Risk, Uncertainty and Decision (RUD) 2014 conference at the University of Warwick for helpful comments.

* Correspondence address: School of Business and Governance, Murdoch University , Australia. Tel.: +61 473646841, +33 6 03 15 83 42; fax: +61 89360 6966
E-mail address: p.blavatskyy@montpellier-bs.com, pavlo.blavatskyy@gmail.com

[1] For example, cumulative prospect theory can account for Allais (1953) common consequence effect, the common ratio effect (*e.g.* Bernasconi, 1994), systematic violations of the betweenness axiom (*e.g.* Camerer and Ho, 1994) and the four-fold pattern of risk attitudes. Schmidt, Starmer, and Sugden (2008) present an extension of the theory that can account for the preference reversal phenomenon as well as the discrepancy between willingness-to-accept and willingness-to-pay. Yet, there are also several behavioral regularities that cumulative prospect theory fails to rationalize such as Blavatskyy (2012b) troika paradox and Machina (2009) reflexion example (see also Blavatskyy, 2013a). Curiously, typical parameterizations of the theory cannot resolve the classical St. Petersburg paradox (Blavatskyy, 2005).

[2] Mathematical expectation of the absolute value of the difference between two realizations of a lottery.

[3] Blavatskyy (2010b) showed that violations of stochastic dominance can be also avoided by measuring financial risks with the mean absolute semideviation (*i.e.* by aggregating only those deviations that are below the expected value).

http://dx.doi.org/10.1016/j.ejor.2016.05.007

using the third L-moment (Hosking, 1990) instead of the third central moment. L-moments are more robust than conventional moments to outliers, which turns out to be sufficient for ruling out violations of stochastic dominance. Note that the first L-moment is the expected value and the second L-moment is simply one half of Gini (1912) mean difference statistic.

The literature on financial decision making uses the model of multi-attribute choice (with attributes being different moments of the distribution of assets' returns). Arguably the simplest decision criterion in multi-attribute choice is to aggregate different attributes into one real-valued index. Such a linear trade-off between the first three L-moments of a lottery represents preferences under Yaari (1987) dual model with a cubic probability weighting function.

A decision maker who prefers positively skewed distributions (*e.g.*, a small chance to win a highly desirable outcome) and dislikes negatively skewed distributions (*e.g.*, a small chance to end up with a highly undesirable outcome) generally has an inverse S-shaped probability weighting function. Moreover, this function crosses the 45° line at a probability smaller (greater) than 0.5 if a decision maker is also averse (attracted) to the dispersion of outcomes. Thus, a well-known probability weighting function empirically discovered by Tversky and Kahneman (1992, p. 309) can be intuitively rationalized as a combination of two factors: an aversion to the second L-moment (dispersion) and an attraction to the third L-moment (skewness). A decision maker not caring about skewness has a simpler probability weighting function—a quadratic polynomial of (de-)cumulative probabilities—that can only be either concave or convex. This probability weighting function is discussed in Delquié and Cillo (2006, pp. 204–205). Yaari (1987) dual model with such a probability weighting function is a special case of the mean-Gini approach of Shalit and Yitzhaki (1984).

Economic decision theory deviated from the idea of risk neutrality by introducing a non-linear (Bernoulli) utility function over money as well as a non-linear probability weighting function over (de-) cumulative probabilities. Financial decision theory deviated from the same idea by introducing a preference for the higher moments of a probability distribution. This paper shows that representing preferences with the first three L-moments is *de facto* equivalent to introducing a cubic probability weighting function. Thus, the two complimentary approaches to modeling decision making under risk from economics and finance can be unified into one general theory.

The remainder of the paper is structured as follows. Cumulative prospect theory is briefly sum-marized in Section 2. Readers familiar with the topic may skip Section 2 without the loss of continuity. A cubic probability weighting function is presented in Section 3. Its relation to statistical L-moments (Hosking, 1990) and mean-Gini approach (Shalit & Yitzhaki, 1984) is discussed in Section 4. Section 5 concludes with a general discussion.

2. Cumulative prospect theory for choice under risk

Let $X \subseteq \mathbb{R}$ denote a nonempty set of possible outcomes (*e.g.* financial returns). A lottery $L{:}X\to[0,1]$ is a discrete probability distribution on set X, *i.e.*, $L(x)\epsilon[0,1]$ for all $x \in X$ and $\Sigma_{x\epsilon X}L(x) = 1$. Any lottery can be alternatively characterized by its cumulative distribution function $F_L{:}X\to[0,1]$. This function gives the probability that lottery L yields an outcome at most as good as outcome $x \in X$:

$$F_L(x) = \sum_{y\in X, x\geq y} L(y) \quad (1)$$

A lottery can be also characterized by its decumulative distribution function $G_L{:}X\to[0,1]$. This function gives the probability that lottery L yields an outcome at least as good as outcome $x \in X$:

$$G_L(x) = 1 - F_L(x) + L(x) \quad (2)$$

In cumulative prospect theory one outcome $r \in X$ is the reference point of a decision maker. Outcomes greater than the reference point are called gains. The set of all gains is denoted by $X_+ \subseteq X$. Outcomes smaller than the reference point are called losses. The set of all losses is denoted by $X_- \subseteq X$.

Preferences of a decision maker are represented by the following utility function:

$$U(L) = \sum_{x\in X_-} [w_-(F_L(x)) - w_-(1 - G_L(x))]u(x) + \sum_{x\in X_+} [w_+(G_L(x)) - w_+(1 - F_L(x))]u(x) \quad (3)$$

where $w_-[0,1]{:}\to[0,1]$ and $w_+[0,1]{:}\to[0,1]$ are two strictly increasing probability weighting functions such that $w_-(0) = w_+(0) = 0$ and $w_-(1) = w_+(1) = 1$; and $u{:}X\to\mathbb{R}$ is an increasing utility function that is unique up to a multiplication by a positive constant and satisfying $u(r) = 0$.[4]

Quiggin (1981) rank-dependent utility is a special case of cumulative prospect theory when either $w_-(p) = 1-w_+(1-p)$ for all $p \in [0,1]$ or all outcomes in X are greater then the reference point r (so that the set of losses X_- is empty). Yaari (1987) dual model is a special case of rank-dependent utility when utility function is linear: $u(x) = x$ for all $x \in X$. Expected utility theory is a special case of rank-dependent utility when a probability weighting function is linear: $w_+(p) = p$ for all $p \in [0,1]$.

3. A cubic probability weighting function

In the following, a probability weighting function is written without subscripts "+" and "−" whenever it is inconsequential whether we deal with gains or losses. We consider a probability weighting function that is a cubic polynomial of probability:

$$w(q) = q - \rho \cdot q(1-q) + \tau \cdot q(1-q)(1-2q) \quad (4)$$

for all $q \in [0,1]$ and two subjective parameters $\rho, \tau\epsilon\mathbb{R}$. Note that function (4) always satisfies $w(0) = 0$ and $w(1) = 1$. Table 1 summarizes the properties of function (4) for various values of parameters ρ and τ.

Fig. 1 plots function (4) for several positive values of parameter τ. Note that the probability weighting function is inverse S-shaped crossing the 45° line at a probability q less than one half when ρ is positive but less then τ (*cf.* dashed curves in Fig. 1). Yet, if ρ is greater than or equal to τ, the probability weighting function does not cross the 45° line at all (*cf.* a solid curve in Fig. 1). When ρ is negative but greater than $-\tau$, the probability weighting function is inverse S-shaped crossing the 45° line at a probability q greater than one half (*cf.* dotted and dashed-dotted curves in Fig. 1). Yet, if ρ is less than or equal to $-\tau$, the probability weighting function does not cross the 45° line at all (*cf.* a dashed-double-dotted curve in Fig. 1). Thus, probability weighting function (4) with a positive value of τ is quite flexible. It can take a variety of shapes including a convex function, a concave function and an inverse S-shaped function crossing the 45° line at various probabilities q.

We can use an existing axiomatization of cumulative prospect theory (with a generic probability weighting function) and impose

[4] There are also additional convexity assumptions. Tversky and Kahneman (1992, p. 305) assumed that both probability weighting functions are inverse S-shaped (concave near probability zero and convex near probability one). This paper relaxes this assumption. Additionally, prospect theory assumes that utility function is convex on X_- and concave on X_+. Finally, the assumption of loss aversion restricts utility function as well (see Köbberling and Wakker, 2005; Blavatskyy, 2011b).