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Decision Support

A new approach to the bi-dimensional representation of the DEA efficient frontier with multiple inputs and outputs

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ABSTRACT

This paper presents a new approach to the graphical presentation of DEA results. Whatever the number of inputs and outputs are, an adequate normalization of their weights is enough to generate a simple bi-dimensional graph, similar to that of the CCR frontier with one input and one output. An advantage over other approaches to the same representation problem is that no complementary techniques are required to plot the frontier. It is also proved that the distance of a DMU to the frontier is related to its efficiency. The proposed approach is also valid for the BCC model.

1. Introduction

In Data Envelopment Analysis (DEA), graphical representations have been used since the seminal paper of Charnes, Cooper, and Rhodes (1978) to show the position of each DMU in relation to the efficient frontier. These representations are powerful support tools for decision makers, for instance to ascertain how far the DMUs are from the efficient frontier, or to look for concentrations of DMUs in some areas in the graph indicating concentration in a market sector.

Initially, bi-dimensional graphical representations in DEA were limited to cases of three variables at most, whether it was a one input-two outputs case or a two inputs-one output case. Later, some researchers looked into graphically representing results of DEA models with more than three variables. However, the techniques proposed so far entail some difficulties, like harder visualization with the increase of variables, use of transformed DEA models which are difficult to interpret, and a lack of clear representation of the efficient frontier, among others, as highlighted in Section 3. In order to avoid these difficulties, this paper proposes a novel bi-dimensional representation of the DEA frontier and the DMUs' location in relation to this frontier, which extends and details the initial suggestion presented by Bana e Costa, Soares de Mello, and Angulo Meza (2014).

In the next section, the standard graphical representations of the efficient frontiers and DMUs are presented. This includes the three variable representation of the CCR model. This is followed by a literature review of different graphical representations and comments in Section 3. Section 4 introduces the proposed bi-dimensional graphical representation and in Section 5 a numerical example is presented. The proposed approach is extended for the BCC model in Section 6. Section 7 illustrates the bi-dimensional graphical representation with real data. In Section 8 we addressed the issue of the multiple optimal weights for efficient DMUs. Finally, Section 9 outlines conclusions of the paper.

2. Standard bi-dimensional DEA representation

Graphical representations for efficiency analysis were used before DEA (Farrell, 1957) to represent the definition of technical efficiency or the theoretical production function or isoquant. When they introduced DEA, Charnes et al. (1978) used two inputs and one output to show the efficient frontier and the location of the DMUs. This was done by transforming the three variables into only two variables, namely input 1 divided by the output and input 2 divided by the output. The CCR fractional model proposed by Charnes et al. (1978), which assumes constant returns to scale

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(CRS), is presented in (1).

$$\text{Max}\frac{\sum_r u_r y_{ro}}{\sum_i v_i x_{io}}$$
s.t.
$$\frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}} \le 1,\ \forall j$$
$$u_r \ge 0,\ v_i \ge 0,\ \forall r, i \quad (1)$$

In this model x_{io} is the input i of DMU o; y_{ro} is the output r of DMU o; v_i is the weight of the input i; u_r is the weight of the output r; x_{ij} is the input i and y_{rj} is the output r of DMU j. The objective function of this formulation is the efficiency index of DMU o, which is the ratio of the virtual output (the weighted sum of the outputs) by the virtual input (the weighted sum of the inputs) of DMU o. This model is run for every DMU in the set.

It should be noted that both the objective function and the constraints are fractional, so it is easy to see (Coelli, Rao, & Battese, 1998; Cooper, Seiford, & Tone, 2007) that if a set of multipliers is an optimal solution for model (1) another solution is obtained with all the multipliers multiplied by the same strictly positive number. This means that model (1) has multiple optimal solutions for all DMUs. In order to obtain only one solution, Charnes et al. (1978) introduced the constraint that the objective function denominator (the weighted sum of the inputs, i.e. the virtual input) equals 1. This is a key point since with such a constraint one also obtains the linearization of model (1). Thus the numerator (the weighted sum of the outputs, i.e. the virtual output) becomes the new objective function and it is equal to the DMU o efficiency index. This formulation is presented in model (2).

$$\text{Max}\sum_r u_r y_{ro}$$
s.t.
$$\sum_i v_i x_{io} = 1$$
$$\sum_r u_r y_{rj} - \sum_i v_i x_{ij} \le 0,\ \forall j$$
$$u_r \ge 0, v_i \ge 0,\ \forall r, i \quad (2)$$

Model (2) provides the efficiency index of the observed DMU (DMU o) which is a real number between 0 and 1; as well as the inputs and outputs weights (also called multipliers) to obtain the value of the efficiency index. This formulation is frequently called the multiplier model. Its dual model, called the envelopment model, provides information about the benchmarks, i.e. the DMU o reference set, and the targets, i.e. inputs and outputs levels for DMU o to become efficient, and of course the efficiency index. The graphical representation of the efficient frontier and the DMUs' location for the one input and one output case is depicted in Fig. 1.

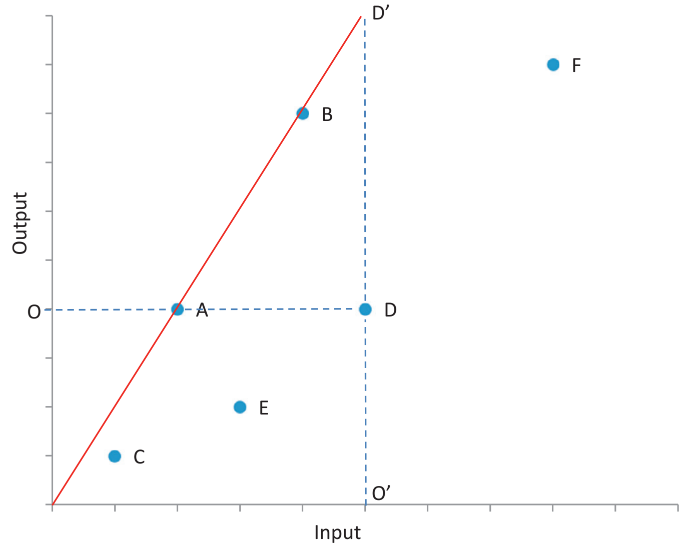

Fig. 1. CCR efficient frontier for one input and one output.

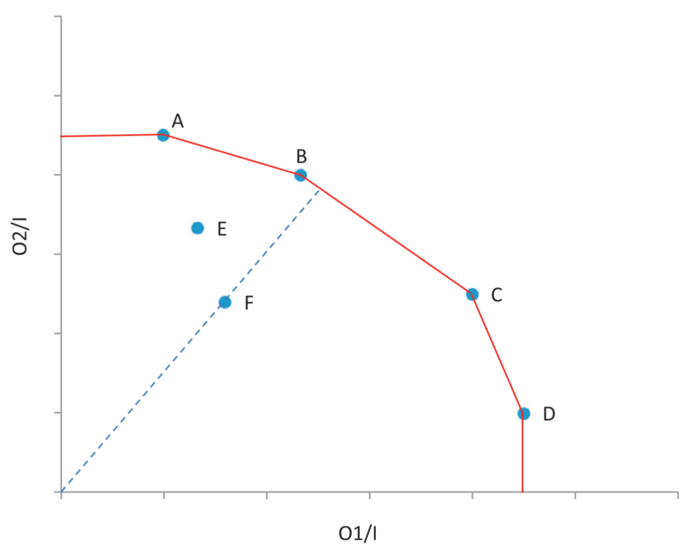

Fig. 2. CRS efficient frontier for one input and two outputs.

In Fig. 1 the efficient frontier is represented by a diagonal that begins at the origin and passes through the most productive DMU or DMUs; DMUs A and B in this example. All DMUs under this frontier are inefficient. Geometrically, the efficiency index is ascertained by measuring the distance of the DMU from the efficient frontier. In the input oriented model, DMU D's efficiency is $\overline{OA}/OD$, which clearly indicates that the output levels are maintained while trying to reduce the inputs used. On the other hand, in the output oriented model, DMU D's efficiency is $\overline{O'D}/\overline{O'D'}$, which indicates that the input levels are maintained while trying to increase the outputs. These measures of efficiency were first devised by Farrell (1957) based on the works by Debreu (1951).

The model presented in (1) and linearized in (2) is input oriented, but in the case of CRS both the input oriented and output oriented efficiency indexes are the same. For details about the output oriented CCR model and respective multipliers and envelopment models, see Cooper, Seiford, and Zhu (2004), for instance.

This standard graphical representation for the CCR model in the plane can also be charted using three variables – two inputs and one output, or one input and two outputs. In the former case, each axis represents each input divided by the output; in the latter, each axis represents each output divided by the input. Fig. 2 depicts the CRS frontier for one input and two outputs and also the projection of inefficient DMU F.

Later, Banker, Charnes, and Cooper (1984) also used a bidimensional representation, for one input and one output, to show a variable returns to scale (VRS) efficient frontier and the differences between the technical and scale efficiencies. The input oriented BCC model is presented in (3) and its linearized formulation is presented in (4).

$$\text{Max}\frac{\sum_r u_r y_{ro} + u_*}{\sum_i v_i x_{io}}$$
s.t.
$$\frac{\sum_r u_r y_{rj} + u_*}{\sum_i v_i x_{ij}} \le 1,\ \forall j$$