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Short Communication

Crash start of interior point methods☆

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ABSTRACT

The starting point used by an interior point algorithm for linear and convex quadratic programming may significantly influence the behaviour of the method. A widely used heuristic to construct such a point consists of dropping variable nonnegativity constraints and computing a solution which minimizes the Euclidean norm of the primal (or dual) point while satisfying the appropriate primal (or dual) equality constraints, followed by shifting the variables so that all their components are positive and bounded away from zero. In this Short Communication a new approach for finding a starting point is proposed. It relies on a few inexact Newton steps performed at the start of the solution process. A formal justification of the new heuristic is given and computational results are presented to demonstrate its advantages in practice. Computational experience with a large collection of small- and medium-size test problems reveals that the new starting point is superior to the old one and saves 20–40% of iterations needed by the primal-dual method. For larger and more difficult problems this translates into remarkable savings in the solution time.

1. Introduction

We are concerned in this paper with the efficient solution of linear and convex quadratic programming problems using interior point methods (IPMs). Such problems are at the heart of many more complicated optimization techniques and progress in their solution impacts the whole optimization area. Following the notation of Gondzio (2012a) we consider the following general primal-dual pair of convex quadratic programming (QP) problems

$$\begin{array}{llll} \text{Primal} & & \text{Dual} & \\ \min & c^Tx+\frac{1}{2}x^TQx & \max & b^Ty-\frac{1}{2}x^TQx \\ \text{s.t.} & Ax=b, & \text{s.t.} & A^Ty+s-Qx=c, \\ & x\geq 0; & & y \text{ free}, \quad s\geq 0, \end{array} \tag{1}$$

where $A\in\mathcal{R}^{m\times n}$ has full row rank $m\leq n$, $Q\in\mathcal{R}^{n\times n}$ is a positive semidefinite matrix, $x,s,c\in\mathcal{R}^n$ and $y,b\in\mathcal{R}^m$. In the special case when $Q=0$ the problems become the pair of primal-dual linear programming (LP) problems.

Standard interior point methods are very sensitive to the choice of a starting point. Many codes use an idea of Mehrotra (1992) and construct a point by solving an auxiliary quadratic programming problem: min $c^Tx+\frac{1}{2}x^TQx+\frac{1}{2}x^Tx$ s.t. $Ax=b$. In this problem all equality constraints are satisfied, but the simple inequalities are ignored. The solution of such a problem may be obtained by an explicit formula at a cost comparable to a single interior point iteration. Since the non-negativity constraint $x\geq 0$ is dropped in it, the solution might contain negative components. Therefore to become an eligible starting point for an IPM, they need to be shifted to positive values. A similar auxiliary problem is formulated to determine an initial dual solution (y,s). Several attempts have been made to improve on this (heuristic) starting point selection and, although some of them offered attractive alternative initialization methods for particular classes of problems, to the best of the author's knowledge, they do not offer a competitive approach for the general case. It is worth mentioning that if a self-dual emdedding (Ye, Todd, & Mizuno, 1994) is used then it is possible to accommodate an arbitrary point and convert it into a starting point (Skajaa, Andersen, & Ye, 2013). However, the implementation of self-dual emdedding needs a slightly more involved linear algebra step (one more back-solve per iteration) and we are not going to use it here.

In this paper we propose a new approach which finds a good initial point for interior point methods applied to a *general* convex quadratic programming problem. We call it a *crash start technique* because it follows a similar principle to that employed by

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simplex solvers and attempts to guess a starting point which is closer to optimality than a default one. Crash start has proved very useful in the context of simplex method for linear programming (Bixby, 1992; Gould & Reid, 1989; Hall, 2010; Maros & Mitra, 1998). Such procedures usually look for an advanced initial basis in which columns corresponding to slack and artificial variables are replaced by those corresponding to structural variables. The heuristics are based on a general expectation that the more those structural columns are inserted into the initial basis the closer it might be to the optimal basis.

The situation in interior point methods is significantly more complicated. Modern IPMs owe their efficiency to the ability to follow closely the central path (Gondzio, 2012a; Wright, 1997). Indeed, both the theory and the computational practice confirm that, as long as the iterates remain in the proximity of the central path, fast progress to optimality can be made. Conversely, if the iterates leave the vicinity of the central path and prematurely approach the boundary of the feasible region, the algorithm might get stuck taking small steps in the Newton direction and the convergence might be disappointingly slow. This means in particular that IPMs cannot be started successfully from an arbitrary point. An ideal initial point should satisfy several requirements:

- it should be close to primal and dual feasibility;
- it should be well centred;
- it should be as close to optimality as possible.

Finding such a point is by no means easy!

In this paper we propose a practical method to construct a point which satisfies all three requirements. Recently there has been a major increase in interest in the use of iterative methods to compute Newton directions in IPMs (D'Apuzzo, De Simone, & Serafino, 2010; Gondzio, 2012a) and a variety of preconditioners for Krylov subspace methods applied in this context have been proposed. Many preconditioners have already been proposed for the normal equations (Schur complement of the KKT system) (Bocanegra, Campos, & Oliveira, 2007; Castro, 2000; Oliveira & Sorensen, 2005) as well as for the indefinite augmented form of the KKT system (Durazzi & Ruggiero, 2003; Gill, Murray, Ponceleón, & Saunders, 1992; Lukšan & Vlček, 1998). There is increasing evidence that using inexact Newton directions (Dembo, Eisenstat, & Steihaug, 1982) in interior point methods is well supported by the theory (Bellavia, 1998; Gondzio, 2013) and works well in practice (Gondzio, 2012a). Our *crash start technique* builds upon these developments.

We observe that at the beginning of the solution process an infeasible interior point method works with large infeasibilities in the primal and dual spaces and a large duality gap. Therefore computing highly accurate Newton directions is not necessary at this stage; very crude inexact directions are able to offer noticeable progress in reducing infeasibilities and the duality gap. Such inexact directions can be computed at a significantly lower cost than exact ones. Consequently we propose to run several initial iterations with directions computed by a preconditioned Krylov subspace method using a very simple (and inexpensive) preconditioner and asking only for very relaxed accuracy to make sure that a few Krylov iterations are enough to deliver an inexact solution. Our choice is a partial Cholesky preconditioner which was designed specially for the matrix-free IPM (Gondzio, 2012b). This preconditioner has several advantages including simplicity and ability to work with very limited (and easy to control) memory requirements.

In our developments in this Short Communication we will follow very closely the recent EJOR survey (Gondzio, 2012a) and therefore we will focus only on several computational aspects which are relevant to the understanding of our crash start approach. Hence this short paper has the following simple structure. In Section 2, we will present the key ideas of interior point methods and in Section 3 we will discuss in detail our new crash start procedure and its implementation. Although our approach to generate an advanced initial solution is only a heuristic, we will provide some simple theoretical justification for it. In Section 4, we will present a comparison of two variants of the interior point method, one using a standard default starting point and one initialized with the proposed crash start solution. Finally, in Section 5 we will give our conclusions.

2. Basics of interior point methods

Path-following interior point methods are well-understood (Gondzio, 2012a; Wright, 1997) and very powerful optimization techniques. An IPM for quadratic programming may be interpreted as an iterative method which follows the path of solutions of the following perturbed first order optimality conditions for (1)

$$\begin{aligned} &Ax = b, \\ &A^T y + s - Qx = c, \\ &XSe = \mu e, \\ &(x, s) \geq 0. \end{aligned} \tag{2}$$

We use a standard IPM notation in which X and S are diagonal matrices in $\mathcal{R}^{n\times n}$ with elements of vectors x and s spread across the diagonal, respectively and $e \in \mathcal{R}^n$ is the vector of ones.

IPMs use the notion of a primal-dual *central path*, being the set of solutions of (2) for any $\mu > 0$. It can be shown that the set of such solutions forms a continuous path $\{(x(\mu), y(\mu), s(\mu)): \mu > 0\}$ and much evidence has been gathered to date that interior point methods benefit from following this path closely (Gonzaga, 1992). In this paper we consider a primal-dual infeasible IPM and therefore define the *symmetric primal-dual infeasible neighbourhood* of the central path as follows

$$\begin{aligned} N_S(\gamma, \beta) = \Big\{ &(x, y, s) \,|\, \|\xi_p\| \leq \frac{\beta\mu}{\mu^0}\|\xi_p^0\|, \\ &\|\xi_d\| \leq \frac{\beta\mu}{\mu^0}\|\xi_d^0\|, \;\; \gamma\mu \leq x_j s_j \leq \frac{1}{\gamma}\mu \Big\}, \end{aligned} \tag{3}$$

where $\xi_p = b - Ax$ and $\xi_d = c - A^T y - s + Qx$ are the violations of primal and dual feasibility constraints, respectively, the superscript zero denotes the initial values of the barrier parameter μ and the infeasibilities ξ_p and ξ_d, $\gamma \in (0, 1)$ controls the width of the neighbourhood and β is a constant. From a computational perspective the most demanding task in IPMs is the computation of the Newton direction $(\Delta x, \Delta y, \Delta s)$ for the nonlinear system (2) which requires solving the following system of linear equations

$$\begin{bmatrix} A & 0 & 0 \\ -Q & A^T & I \\ S & 0 & X \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} \xi_p \\ \xi_d \\ \xi_\mu \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - s + Qx \\ \sigma\mu e - XSe \end{bmatrix}. \tag{4}$$

The parameter $\sigma \in (0, 1)$ controls the aspiration of how much one would like to reduce the barrier term $\mu^{k+1} = \sigma\mu^k$. Given the Newton direction, a maximum stepsize α which keeps the new iterate $(\bar{x}, \bar{y}, \bar{s}) = (x, y, s) + \alpha(\Delta x, \Delta y, \Delta s)$ in the neighbourhood (3) is determined and then the algorithm makes this step to a new iterate. We summarize the algorithm below.