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Discrete Optimization

Cutting plane algorithms for 0-1 programming based on cardinality cuts

Osman Oguz *

Department of Industrial Engineering, Bilkent University, Ankara, Turkey

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ABSTRACT

We present new valid inequalities for 0-1 programming problems that work in similar ways to well known cover inequalities. Discussion and analysis of these cuts is followed by their revision and use in integer programming as a new generation of cuts that excludes not only portions of polyhedra containing noninteger points, also parts with some integer points that have been explored in search of an optimal solution. Our computational experimentations demonstrate that this new approach has significant potential for solving large scale integer programming problems.

1. Introduction

Branch-and-cut was introduced in [2] demonstrating the important role of the use of Gomory cutting planes [4] and cover inequalities in the branch-and-bound process for solving integer programming problems. Relatively recent works like [5,6] provide extensive discussions of available strategic choices for using cover inequalities in the branch-and-cut process for 0-1 programming. One may see [7,10] for basic expositions of the subject and related issues.

We work on the 0-1 integer programming problem given below to introduce new valid inequalities similar to cover and lifted cover inequalities. We have chosen this problem to introduce our approach, because most of the work on cover inequalities are based on this problem. As a good example, we can mention [3] that describes an implementation of cover cuts on the multiple knapsack version of the problem.

$$\textbf{IP}\quad \text{Maximize}\quad z=\sum_{j=1}^{n} c_j x_j \tag{1}$$

$$\text{subject to}\quad \sum_{j=1}^{n} a_{ij}x_j \leqslant b_i \quad \text{for } i=1,\dots,m, \tag{2}$$

$$x_j \in \{0,1\} \quad \text{for } j=1,\dots,n, \tag{3}$$

m and n are the number of constraints and decision variables, respectively.

We do not assume any restrictions the integrality or nonintegrality of the parameters c_j, a_{ij} and b_i.

The next section consists of the description and the generation method of the inequalities together with the proof of validity. Section 3 is devoted to redefining and improving the performance of the proposed cuts. The preliminary numerical experiments are discussed in Section 4. Conclusions and comments follow in Section 5.

2. The new cut

Consider the problem **IP** and let $X_{\text{LP}} = (x_1,\dots,x_n)$ denote a solution to the linear programming (LP) relaxation of this problem. Also let $S_p = \{j|x_j > 0 \text{ in } X_{\text{LP}}\}$ and solve the following problem:

$$z_0 = \max \sum_{j=1}^{n} x_j : \sum_{j=1}^{n} a_{ij}x_j \leqslant b_i \quad \text{for } i=1,\dots,m \quad \text{and } x_j \in [0,1] \quad \text{for } j=1,\dots,n. \tag{4}$$

The following inequality is obviously valid:

$$\sum_{j\in S_p} x_j \leqslant z_0. \tag{5}$$

Also, this inequality is valid for all possible values x_j in any solution of the LP relaxation for any objective function. Moreover, the inequality is valid in the form,

$$\sum_{j\in S_p} x_j \leqslant \lfloor z_0 \rfloor \tag{6}$$

for any integer solution of the problem for any objective function. In fact, this last inequality may be an effective cut to make some noninteger solutions infeasible. However, its use can be limited to very few instances and it becomes ineffective very soon in a cutting plane framework. It may even be useless if z_0 is integer valued or $\sum_{j\in S_p} x_j \leqslant \lfloor z_0 \rfloor$ is not violated by the relaxed solution. Nonetheless, it is the starting point of our proposal for a new type of cuts.

* Tel.: +90 312 290 1544; fax: +90 312 266 4054.
E-mail address: ooguz@bilkent.edu.tr

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Starting with the solution to the LP relaxation of the problem **IP**, we partition N, the index set of variables of **IP** into two subsets: $S_1 = \{j x_j = 1 \text{ in } X_{LP}\}$ and $S_2 = N \setminus S_1$. Then, naming the LP relaxation of the original integer program **IP** as **P**, we define the following linear program named as **P1**:

$$\textbf{P1} \quad z_1 = \text{maximize} \quad \sum_{j \in S_2} x_j \tag{7}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i \quad \text{for } i = 1, \ldots, m, \tag{8}$$

$$\sum_{j \in S_1} x_j = |S_1|, \tag{9}$$

$$0 \leqslant x_j \leqslant 1 \quad \text{for } j = 1, \ldots, n. \tag{10}$$

Then the following is true.

Proposition 1. *The inequality*

$$\sum_{j \in S_1} r x_j + \sum_{j \in S_2} x_j \leqslant r|S_1| + \lfloor z_1 \rfloor \tag{11}$$

is valid for the solution set of the problem **IP** *for* $r = |S_2|$.

Proof. It is obvious that the above inequality is violated by the current X_{LP} if $\sum_{j \in S_2} x_j > \lfloor z_1 \rfloor$. On the other hand, the inequality is valid for all integer solutions satisfying the condition $\sum_{j \in S_2} x_j \leqslant \lfloor z_1 \rfloor$. However, when $\sum_{j \in S_2} x_j > \lfloor z_1 \rfloor$ holds for an integer solution, that solution must have at least one x_j with $j \in S_1$ equal to zero in order that the solution is feasible, by the definition of z_1 in **P1**. Thus we conclude that the number of x_j's for $j \in S_2$ being equal to 1 can be greater than $\lfloor z_1 \rfloor$, only if at least one variable x_j in S_1 is equal to 0. Setting $r = |S_2|$ will allow the inequality to hold even when all variables in S_2 are equal to 1, and only one variable in S_1 is equal to 0. □

The valid inequality of Eq. (11) will be called the cardinality cut.

Assigning large values to the parameter ***r*** is not desirable since the quality of the cut will be poor, i.e., the cut will remove a relatively small part of the underlying polyhedron containing no integer solutions. A more reasonable approach is to choose ***r*** more carefully. Consider a slight variation of **P1** given below:

$$\textbf{P2} \quad z_2 = \text{maximize} \sum_{j \in S_2} x_j \tag{12}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i \quad \text{for } i = 1, \ldots, m, \tag{13}$$

$$\sum_{j \in S_1} x_j = |S_1| - 1, \tag{14}$$

$$0 \leqslant x_j \leqslant 1 \quad \text{for } j = 1, \ldots, n. \tag{15}$$

Assuming $z_2 > z_1$, without loss of generality, $z_2 - \lfloor z_1 \rfloor$ is an upperbound on how many more x_j's in S_2 can take a value of 1 when we decrease the cardinality of S_1 by 1. Note that when we replace the right hand side of the equation $\sum_{j \in S_1} x_j = |S_1| - 1$ by $|S_1| - 2$, the difference $z_2 - \lfloor z_1 \rfloor$ will less than double, since z_2 parametrized by $|S_1| - k$, is a concave piecewise linear function for $k \geqslant 0$. Thus, setting $r = z_2 - \lfloor z_1 \rfloor$ gives a relaxation sufficiently tight for the purpose of cutting deeper into the underlying polyhedron.

There will, of course, be instances ($z_1 = \lfloor z_1 \rfloor$ for example), such that the valid inequality will fail to eliminate X_{LP} . We can try a few more things before giving up and starting branching. The most obvious thing to do is to play around the partition of N into S_1 and S_2. We have tried two strategies with partial success. First one is to move few variables from S_2 to S_1 picking those with values close to 1. Second strategy is to eliminate some variables in S_2 from consideration, i.e., not including them in the valid inequalities, or in the objective function of problem **P2**. We may end up with effective cuts as a result of these changes. Second strategy and its variations seem to be working better in our preliminary experimentations.

We report results comparing the efficiency of these new cuts with that of the cover inequalities in a cutting plane framework on a set of hard multidimensional knapsack problems described in [8,9]. Note that, the new cuts may be used for the traveling salesman problem, set packing or covering problems, and other **NP-*Hard*** problems with 0-1 constraint matrices as well, without any adaptation of the inequality given in Eq. (11). This is an extra feature of the new cut over the capability of ordinary cover inequalities.

Although the comparison mentioned above indicates superior efficiency of the cardinality cuts over the cover inequalities, we have discovered that their functionality and efficiency may be further enhanced by slightly changing their definition and using them in a novel algorithmic approach. This led to some significant improvements for the solution of large scale 0-1 integer programming problems. The next section reports these developments.

3. Redefinition, optimization and aggregation of the cardinality cuts

The inequality given by Eq. (11) is a lifted version of the inequality of Eq. (6). We take a further step in this direction and obtain what might be called the overlifted version of Eq. (11), because lifting is done for the purpose of eliminating a certain set of feasible integer solutions from the solution space.

Let us consider the following revised version of **P1** defined by Eqs. (7)–(10):

$$\textbf{P3} \quad z_3 = \text{maximize} \quad \sum_{j \in S_2} c_j x_j \tag{16}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i \quad \text{for } i = 1, \ldots, m, \tag{17}$$

$$\sum_{j \in S_1} x_j = |S_1|, \tag{18}$$

$$x_j \in \{0, 1\} \quad \text{for } j = 1, \ldots, n. \tag{19}$$

The optimal solution of this problem is a feasible solution for **IP**. Also, $\text{ZINT}_{LB} = z_3 + \sum_{j \in S_1} c_j$ is a lower bound for the optmal objective function value of **IP**. We call the following version of the cardinality cut "the optimized cardinality cut":

$$\sum_{j \in S_1} r x_j + \sum_{j \in S_2} x_j \leqslant r|S_1| \tag{20}$$

with $r = |S_2|$.

Then, we state and prove the following proposition:

Proposition 2. *The optimized cardinality cut represented by the inequality in Eq. (20), when added to* **P**, *makes sure that all solutions of* **P3**, *except for the solution* $X = \{j | x_j = 1 \text{ for } j \in S_1 \text{ and } x_j = 0 \text{ for } j \in S_2\}$, *are infeasible while all other integer solutions of* **IP** *remain feasible.*

Proof. It is obvious that for any positive value of r, the inequality will be violated when any variable x_j for $j \in S_2$ takes a positive value while all $x_j = 1$ for $j \in S_1$. On the other hand, setting only one $x_j = 0$ such that $j \in S_1$ will make room for all x_j's such that $j \in S_2$ to take positive values if we set $r = |S_2|$. □

Corollary 1. *When the inequality of Eq. (20) is added to the constraint set of* **P**, *the LP relaxation of* **IP**, *with* $r = |S_2|$, *it separates only the integer and noninteger solutions having all* $x_j = 1$ *with* $j \in S_1$ *as a proper subset of variables with nonzero values, from the feasible set of* **P**. *All other integer feasible solutions remain in the feasible set.*