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Continuous Optimization

A two-phase algorithm for the multiparametric linear complementarity problem

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ABSTRACT

A new two-phase method for solving the multi-parametric linear complementarity problem (mpLCP) with sufficient matrices is presented. In the first phase an initial feasible solution to mpLCP which satisfies certain criteria is determined. In the second phase the set of feasible parameters is partitioned into polyhedral regions such that the solution of the mpLCP, as a function of the parameters, is invariant over each region. The worst-case complexity of the presented algorithms matches that of current methods for nondegenerate problems and is lower than that of current methods for degenerate problems.

1. Introduction

We consider a parametric form of the Linear Complementarity Problem (LCP) in which the right hand side vector is dependent on a vector of parameters $\theta \in S_\theta \subseteq \mathbb{R}^k$, where S_θ is a bounded convex polytope defining the set of "attainable" values for θ. This problem, referred to as the multiparametric Linear Complementarity Problem (mpLCP), is as follows:

Given $M \in \mathbb{R}^{h\times h}$, $q \in \mathbb{R}^h$ and $\triangle Q \in \mathbb{R}^{h\times k}$ $(k \leq h)$, for each $\theta \in S_\theta$ find vectors $w(\theta)$ and $z(\theta)$ in $\mathbb{R}^h$ that satisfy the following system:

$$\begin{aligned} w - Mz &= q + \triangle Q\theta \\ w^\top z &= 0 \\ w, z &\geq 0 \end{aligned} \tag{1}$$

If such a solution exists for a given $\theta \in S_\theta$, mpLCP is said to be *feasible at* θ, otherwise it is *infeasible at* θ. Similarly, mpLCP is said to be *feasible* if there exists a $\hat{\theta} \in S_\theta$ at which mpLCP is feasible, otherwise mpLCP is *infeasible*. As finding a solution to (1) for each $\theta \in S_\theta$ individually is intractable, the goal of mpLCP is to partition the space S_θ into regions such that the representation of the solution vectors w and z as functions of θ is invariant over each region. In the literature these regions have been given a variety of names, such as invariancy regions, critical regions, and validity sets. We refer to them as *invariancy regions* and discuss them in more detail in the next section.

LCP, and by extension mpLCP, has numerous applications in the fields of engineering and economics. For an extensive list we suggest (Cottle, Pang, & Stone, 2009; Murty & Yu, 1997). It is well known that Linear Programs (LPs) Quadratic Programs (QPs) with convex objective functions and linear constraints can be reformulated as LCPs. Thus, mpLCP encompasses multiparametric LPs (mpLPs) and multiparametric QPs (mpQPs) containing parameters in the linear term of the objective function and in the right hand sides of the constraints. Recently mpQPs of this form have received much attention in the literature for their application to model predictive control (Baotić, 2002; Bemporad, Morari, Dua, & Pistikopoulos, 2000; Grancharova & Johansen, 2012; Gupta, Bhartiya, & Nataraj, 2011; Patrinos & Sarimveis, 2010; Pistikopoulos, Dua, Bozinis, Bemporad, & Morari, 2002; Spjøtvold, Kerrigan, Jones, Tøndel, & Johansen, 2006; Spjøtvold, Tøndel, & Johansen, 2007; Tøndel, Johansen, & Bemporad, 2003a; 2003b).

Another important class of problems that has received considerable attention in recent years and can also be formulated as a mpQP is multiobjective optimization problems with a single psuedoconvex objective and any number of linear objectives. These types of problems are particularly relevant in the areas of economics and finance. Examples of works considering these types of problems include Hirschberger, Qi, and Steuer (2010), Ponsich, Jaimes, & Coello (2013), Smimou (2014), Yu & Lee (2011), Zopounidis, Galariotis, Doumpos, Sarri, & Andriosopoulos (2015) and the references therein.

In general LCP is NP-hard, though polynomial time algorithms exist for certain classes of the matrix M. Thus, much work has been done in order to identify various classes of matrices M which impact one's ability to solve an instance of LCP. Solution techniques

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for LCP are often designed for specific classes of M. For a concise list of important matrix classes see Cottle (2010). For a detailed discussion on these classes and their impact on LCP see Cottle et al. (2009); Murty and Yu (1997). We will refer to many of the matrix classes discussed in these works throughout this paper. As the method we proposed requires that M be a sufficient matrix, we provide the following definition, as found in Cottle et al. (2009).

Definition 1.1. A matrix $M \in \mathbb{R}^{h\times h}$ is *column sufficient* if the following implication is satisfied:

$$\{x_i(Mx)_i \le 0 \text{ for all } i\} \quad \Rightarrow \quad \{x_i(Mx)_i = 0 \text{ for all } i\} \tag{2}$$

M is said to be *row sufficient* if $M^\top$ is column sufficient. If M is both column and row sufficient, it is then called *sufficient*.

Parametric LCP with a single parameter (i.e., $k = 1$) has been studied quite extensively. Some of the works considering this problem include Cottle (1972), Danao (1997), Pang (1980) and Pang, Kaneko, and Hallman, (1979). Columbano, Fukuda, and Jones (2009), Gailly, Installe, and Smeers (2001), Jones and Morari (2006) and Li and Ierapetritou (2010) consider mpLCP as in (1) (i.e., $k > 1$). The method of Gailly et al. (2001) is designed for the case in which M is copositive-plus. The method is theoretically sound but lacks a practical discussion as to how the theory should be implemented. Jones and Morari (2006) propose a method for the case in which M is positive semi-definite. Their method is an extension of techniques that are used for solving single parametric LCP, but depends on a lexicographic ϵ-perturbation in order to handle degeneracy. Columbano et al. (2009) developed a technique for instances in which M is a sufficient matrix. When certain conditions are not satisfied, however, their method depends on an ϵ-perturbation technique in which an auxiliary multiobjective program must be solved. The method of Li and Ierapetritou (2010) works for general M, but is computationally expensive since it requires reformulating the mpLCP as a multiparametric bilinear mixed integer program. Recently, Herceg, Jones, Kvasnica, and Morari (2015) proposed a technique designed for general M which extends the enumerative approach of Gupta et al. (2011) for solving mpQP to the context of mpLCP.

Significant improvements can still be made on solution techniques for mpLCP. In this paper we propose a two-phase technique for solving instances of mpLCP in which M is sufficient. Phase 1 is used for initialization and only terminates when: (i) an instance of mpLCP has been shown to be infeasible, or (ii) an initial feasible solution and the corresponding invariancy region have been discovered. In the latter case, Phase 2 is then used to partition S_θ. Phase 2 is inspired by the work of Columbano et al. (2009), but does not rely on an ϵ-perturbation technique and therefore has an improved worst-case complexity. We point out that in our consideration of Phase 1 we answer a very important question that no other work we are aware of has considered, the question of how one can determine an initial feasible solution for a (multi)parametric LCP problem. In all works we know of, it is simply assumed that such a solution is available.

As mentioned, the method for solving mpLCP which we present in this work is a two-phase method. We will show that the problem solved in the first phase of this method is a special case of the problem solved during the second phase. For this reason we discuss Phase 2 prior to Phase 1. Hence, the remainder of this work is organized as follows. Background information on LCP problems and their geometrical structure is contained in Section 2. The theory and methodology for Phase 2 of the proposed method for solving mpLCP are presented in Section 3. In Section 4 we present the theory and methodology for Phase 1. We discuss the complexity of each algorithm and present numerical results for applying the proposed two-phase method to a collection of mpQP instances in Section 5. In Section 6 we provide concluding remarks and a discussion on proposed future work. In Appendix A we offer an illustrative example, showing explicitly how the Phase 1 and 2 algorithms are used to solve an instance of mpLCP. Appendix B contains detailed results from our computational experiments as well as a couple of supporting images.

2. Background on mpLCP

This section is divided into two subsections. In the first we present preliminary notations and definitions and in the second we provide a discussion on the geometry of mpLCP and provide some preliminary results.

2.1. Preliminaries

We begin this subsection by introducing definitions and notation necessary for the remainder of this work. Assume that we are given an mpLCP of the form (1) and define the matrix $G := [I \ \ -M]$ and the vector $\nu := \begin{bmatrix} w \\ z \end{bmatrix}$, where $G \in \mathbb{R}^{h\times 2h}$ and $\nu \in \mathbb{R}^{2h}$. We use the notation $G_{i\bullet}$ to represent the i^{th} row of G and $G_{\bullet j}$ to represent the j^{th} column of G. Also, given a set $\mathcal{I} \subseteq \{1, \dots, h\}$ we use $G_{\mathcal{I}\bullet}$ to denote the matrix formed by the rows of G indexed by $\mathcal{I}$. Similarly, given a set $\mathcal{J} \subseteq \{1, \dots, 2h\}$ we use $G_{\bullet\mathcal{J}}$ to denote the matrix formed by the columns of G indexed by $\mathcal{J}$. Furthermore, given $\mathcal{I} \subseteq \{1, \dots, h\}$ and $\mathcal{J} \subseteq \{1, \dots, 2h\}$, we use $G_{\mathcal{I}\mathcal{J}}$ to represent the submatrix of G consisting of the elements of the rows indexed by $\mathcal{I}$ which are in the columns indexed by $\mathcal{J}$, i.e., $G_{\mathcal{I}\mathcal{J}} = (G_{\mathcal{I}\bullet})_{\bullet\mathcal{J}}$. Let $\mathcal{E}$ denote the index set $\{1, \dots, 2h\}$ for (1).

Definition 2.1. A *basis* is a set $\mathcal{B} \subset \mathcal{E}$ such that $|\mathcal{B}| = h$ and $rank(G_{\bullet\mathcal{B}}) = h$. The set $\mathcal{N} := \mathcal{E} \setminus \mathcal{B}$ is called the *complement* of $\mathcal{B}$.

Definition 2.2. The sets of variables $\nu_{\mathcal{B}} := \{\nu_i : i \in \mathcal{B}\}$ and $\nu_{\mathcal{N}} := \{\nu_i : i \in \mathcal{N}\}$ are referred to as the sets of *basic* and *nonbasic* variables, respectively.

Definition 2.3. Given a basis $\mathcal{B}$, for every $\theta \in S_\theta$, $\nu_{\mathcal{B}}(\theta) = G^{-1}_{\bullet\mathcal{B}}(q + \triangle Q\theta)$, $\nu_{\mathcal{N}}(\theta) = 0$ is a solution to the linear system: $G\nu = q + \triangle Q\theta$. For each $\theta \in S_\theta$, the solution $\big(\nu_{\mathcal{B}}(\theta), \nu_{\mathcal{N}}(\theta)\big)$ is called a *basic solution*.

Definition 2.4. A basis $\mathcal{B}$ is called *complementary* if $|\{i, i+h\} \cap \mathcal{B}| = 1$ for each $i \in \{1, \dots, h\}$.

We have now built the tools necessary for providing the definition of an *invariancy region*. Consider a complementary basis $\mathcal{B}$ and suppose there exists $\theta \in S_\theta$ such that: (i) $\nu_{\mathcal{B}}(\theta) = G^{-1}_{\bullet\mathcal{B}}(q + \triangle Q\theta) \ge 0$ and, (ii) $\nu_{\mathcal{N}}(\theta) = 0$. Then since $\nu = \begin{bmatrix} w \\ z \end{bmatrix}$, for all $\theta \in S_\theta$ satisfying (i) and (ii) above, the basic solution $\big(\nu_{\mathcal{B}}(\theta), \nu_{\mathcal{N}}(\theta)\big)$ satisfies (1) and therefore defines solution vectors $w(\theta)$ and $z(\theta)$ for mpLCP. Note that one set of solution vectors of this form may exist for each complementary basis.

Definition 2.5. The *invariancy region* $\mathcal{IR}_{\mathcal{B}}$ of a complementary basis $\mathcal{B}$ is the set:

$$\mathcal{IR}_{\mathcal{B}} := \left\{\theta \in S_\theta \subseteq \mathbb{R}^k : G^{-1}_{\bullet\mathcal{B}}(q + \triangle Q\theta) \ge 0\right\} \tag{3}$$

Hence, there may exist one invariancy region for each complementary basis.

Definition 2.6. A complementary basis $\mathcal{B}$ is called *feasible* to (1) if $\mathcal{IR}_{\mathcal{B}} \neq \emptyset$.

Every invariancy region is a convex polytope contained within S_θ. For every feasible complementary basis $\mathcal{B}$, the affine function defined by $\nu_{\mathcal{B}}(\theta) = G^{-1}_{\bullet\mathcal{B}}(q + \triangle Q\theta)$, $\nu_{\mathcal{N}}(\theta) = 0$ is a solution to (1) for all $\theta \in \mathcal{IR}_{\mathcal{B}}$. Therefore in this work we propose a method