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Continuous Optimization

Multitime multiobjective variational problems and vector variational-like inequalities[☆]

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ABSTRACT

In this paper, we introduce vector variational-like inequality with its weak formulation for multitime multiobjective variational problem. Moreover, we establish the relationships between the solutions of introduced inequalities and (properly) efficient solutions of multitime multiobjective variational problem, involving the invexities of multitime functionals. Some examples are provided to illustrate our results.

1. Introduction

In optimization problems, one investigates the best solutions from all feasible solutions. These kinds of problems with two or more objective functions to be optimized simultaneously are called vector optimization problems. Maximizing profit and minimizing the cost of a product, maximizing performance and minimizing the fuel consumption of a vehicle are some examples of vector optimization problems. It is difficult to find unique solution of these types of problems because they rarely have feasible points that simultaneously maximize or minimize all the objectives. Hence, it is necessary to explore the concept of efficient solutions. Several authors and researchers have devoted in this direction. Borwein (1983) derived the existence conditions for the efficient set in real linear spaces with respect to ordering induced by convex cones and Kazmi (1996) proved the existence of a weak minimum for constrained vector optimization problems. Similar results have been established by Sergienko, Lebedeva, and Semenova (2000) for the efficient solutions based on the properties of recession cones of convex feasible sets.

Convexity is being used to study a large number of problems arising in management sciences, engineering and economics. It is the most inevitable hypothesis in optimization theory and used to obtain the sufficient condition, for that condition, which is only necessary. The growing interest in optimization, asks for the generalization of convexity, as the concept of convexity does no longer suffices in real world problems. Initially, Hanson (1981) generalized convex functions to introduce the concept of invexity. There are some other generalizations such as preinvex, univex, pseudoinvex, approximate convex functions etc. For more contributions see, Khurana (2005) and Gupta, Mehra, and Bhatia (2006).

Variational inequalities have wide applications in traffic analysis, game theory, physics, mechanics and so on because these problems can be transformed into variational inequalities. Due to these extensive attentions vector variational inequality was developed and initially formulated by Giannessi (1980). Vector variational inequalities are efficient tool for the investigation of vector optimization problems because these inequalities ensure the existence of efficient solutions, under the condition of convexity or generalized convexity. Many works of these type of inequalities have been focused on looking for the relations between the solutions of various type of vector variational inequalities and vector optimization problems. For various approaches, we refer to Ruiz-Garzón, Osuna-Gómez, and Rufián-Lizana (2004), Al-Homidan and Ansari (2010), Oveisiha and Zafarani (2013), Jayswal, Choudhury, and Verma (2014), and Zegeyea and Shahzadb (2014).

Variational and optimal control problems proved to be a very useful and powerful tool for the study in engineering problems, for instance, control design for autonomous vehicles, economics, operations research. Variational problems are divided into

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two categories: one is vector continuous-time problem and other is classical variational problem. Kim (2004) formulated vector variational-type inequalities and demonstrated the relationships between these inequalities and vector continuous-time problems. Postolache (2012) has contributed by proving some duality theorems for variational problems of minimizing a vector of quotients of functionals of curvilinear integral type. Pitea and Antczak (2014) considered a new class of generalized nonconvex multitime multiobjective variational problem and proved sufficient optimality conditions for efficient and proper efficient solutions. Other approaches have been well documented in Arana-Jiménez, Ruiz-Garzón, Rufián-Lizana, and Osuna-Gómez (2010), Pitea and Postolache (2012) and Ruiz-Garzón, Osuna-Gómez, Rufián-Lizana, and Hernández-Jiménez (2015).

In continuation of these, we present our paper, in which we introduce (weak) vector variational-like inequality for multitime multiobjective variational problems. The organization of the paper is as follows. In Section 2, we recall some preliminaries and definitions, which are helpful in proving our results. Relations between the solutions of introduced inequalities and multitime multiobjective variational problem are derived in Section 3. Ultimately, Section 4, concludes our paper.

2. Notations and preliminaries

Let M and N be Riemannian manifolds of the dimensions m, n, endowed with local coordinates $t = t^\alpha, \alpha = \{1, \dots, m\}$ and $x = x^j, j = \{1, \dots, n\}$, respectively. By using the product order relation on $\mathbb{R}^m$, the hyperparallelepiped $\beta_{t_\circ,t_1}$ in $\mathbb{R}^m$ with diagonal opposite points $t_\circ = (t_\circ^1, \dots, t_\circ^m)$ and $t_1 = (t_1^1, \dots, t_1^m)$ can be written as interval $[t_\circ, t_1]$. Further, let $\Gamma_{t_\circ,t_1}$ be a piecewise C^1-class curve joining the points $t_\circ$, t_1 and $J^1(M, N)$ be the first order jet bundle associated to M and N. $C^\infty(\beta_{t_\circ,t_1}, N)$ denotes the space of functions $x : \beta_{t_\circ,t_1} \mapsto N$ of C^∞-class with the norm

$$\|x\| = \|x\|_\infty + \sum_{\alpha=1}^{m} \|x_\alpha\|_\infty.$$

From now onwards let $X \subset C^\infty(\beta_{t_\circ,t_1}, N)$ unless otherwise specified. For any n-dimensional vectors x and y, we use the following convention for equalities and inequalities throughout the paper:

(a) $x \leq y \Leftrightarrow x_i \leq y_i, i = 1, ..., n,$ with strict inequality holding for at least one i;
(b) $x \leqq y \Leftrightarrow x_i \leqq y_i, i = 1, ..., n$;
(c) $x = y \Leftrightarrow x_i = y_i, i = 1, ..., n$;
(d) $x < y \Leftrightarrow x_i < y_i, i = 1, ..., n$.

Consider the following multitime multiobjective variational problem:

(MVP) Minimize $\int_{\Gamma_{t_\circ,t_1}} f_\alpha(\pi_x(t))dt^\alpha$

$$= \left(\int_{\Gamma_{t_\circ,t_1}} f_\alpha^1(\pi_x(t))dt^\alpha, \dots, \int_{\Gamma_{t_\circ,t_1}} f_\alpha^p(\pi_x(t))dt^\alpha \right)$$

subject to $x(t) \in X$,

where $f_\alpha^i : J^1(M, N) \mapsto \mathbb{R}$, $i \in P = \{1, \dots, p\}$ is closed 1-form of C_∞-class and $\pi_x(t) = (t, x(t), x_\gamma(t))$, $x_\gamma(t) = \frac{\partial x(t)}{\partial t^\gamma}$, $\gamma = \{1, \dots, m\}$ are partial velocities.

The closeness conditions (complete integrability conditions) are

$$D_\alpha f_\beta^i = D_\beta f_\alpha^i, \ i \in P, \ \alpha = \beta = \{1, \dots, m\} \text{ and } \alpha \neq \beta,$$

where D_α and D_β are total derivatives.

Multitime multiobjective variational problems have an inevitable deal, finding (weakly, properly) efficient solutions from the set of all feasible solutions in the following sense:

Definition 2.1 (Pitea & Antczak, 2014). A point $y(t) \in X$ is called an efficient solution of (MVP), if there exists no $x(t) \in X$ such that

$$\int_{\Gamma_{t_\circ,t_1}} f_\alpha^i(\pi_x(t))dt^\alpha - \int_{\Gamma_{t_\circ,t_1}} f_\alpha^i(\pi_y(t))dt^\alpha \leq 0, \quad \forall\, i \in P,$$

with strict inequality for at least one i.

Definition 2.2 (Pitea & Antczak, 2014). A point $y(t) \in X$ is called a weak efficient solution of (MVP), if there exists no $x(t) \in X$ such that

$$\int_{\Gamma_{t_\circ,t_1}} f_\alpha^i(\pi_x(t))dt^\alpha - \int_{\Gamma_{t_\circ,t_1}} f_\alpha^i(\pi_y(t))dt^\alpha < 0, \quad \forall\, i \in P.$$

Definition 2.3 (Pitea & Antczak, 2014). A point $y(t) \in X$ is said to be a proper efficient solution of (MVP), if it is an efficient solution of (MVP) and if there exists a positive scalar M such that for all $i \in P$,

$$\int_{\Gamma_{t_\circ,t_1}} f_\alpha^i(\pi_y(t))dt^\alpha - \int_{\Gamma_{t_\circ,t_1}} f_\alpha^i(\pi_x(t))dt^\alpha$$
$$\leq M\left(\int_{\Gamma_{t_\circ,t_1}} f_\alpha^j(\pi_x(t))dt^\alpha - \int_{\Gamma_{t_\circ,t_1}} f_\alpha^j(\pi_y(t))dt^\alpha \right),$$

for some j such that

$$\int_{\Gamma_{t_\circ,t_1}} f_\alpha^j(\pi_x(t))dt^\alpha > \int_{\Gamma_{t_\circ,t_1}} f_\alpha^j(\pi_y(t))dt^\alpha,$$

whenever, $x(t) \in X$ and

$$\int_{\Gamma_{t_\circ,t_1}} f_\alpha^i(\pi_x(t))dt^\alpha < \int_{\Gamma_{t_\circ,t_1}} f_\alpha^i(\pi_y(t))dt^\alpha.$$

Now, we introduce the following vector variational-like inequality and weak vector variational-like inequality, respectively, which will be used to ensure the existence of efficient solutions of considered multitime multiobjective variational problem (MVP) in sequel of the paper.

Let $f_\alpha^i : J^1(\beta_{t_\circ,t_1}, N) \mapsto \mathbb{R}$, $i \in P$ is closed 1-form of C_∞-class, $\eta : J^1(\beta_{t_\circ,t_1}, N) \times J^1(\beta_{t_\circ,t_1}, N) \mapsto \mathbb{R}^n$ and D_γ is total derivative.

(VVLI) Find $y(t) \in X$ such that there exists no $x(t) \in X$, satisfying

$$\Bigg(\int_{\Gamma_{t_\circ,t_1}} \Bigg[\left\langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_\alpha^1}{\partial x}(\pi_y(t)) \right\rangle + \left\langle D_\gamma \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_\alpha^1}{\partial x_\gamma}(\pi_y(t)) \right\rangle \Bigg] dt^\alpha, \dots,$$
$$\int_{\Gamma_{t_\circ,t_1}} \Bigg[\left\langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_\alpha^p}{\partial x}(\pi_y(t)) \right\rangle + \left\langle D_\gamma \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_\alpha^p}{\partial x_\gamma}(\pi_y(t)) \right\rangle \Bigg] dt^\alpha \Bigg) \leq 0.$$

(WVVLI) Find $y(t) \in X$ such that there exists no $x(t) \in X$, satisfying

$$\Bigg(\int_{\Gamma_{t_\circ,t_1}} \Bigg[\left\langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_\alpha^1}{\partial x}(\pi_y(t)) \right\rangle + \left\langle D_\gamma \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_\alpha^1}{\partial x_\gamma}(\pi_y(t)) \right\rangle \Bigg] dt^\alpha, \dots,$$
$$\int_{\Gamma_{t_\circ,t_1}} \Bigg[\left\langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_\alpha^p}{\partial x}(\pi_y(t)) \right\rangle + \left\langle D_\gamma \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_\alpha^p}{\partial x_\gamma}(\pi_y(t)) \right\rangle \Bigg] dt^\alpha \Bigg) < 0.$$

Following example shows that, vector variational-like inequality (VVLI), which we have introduced is solvable at a point.