European Journal of Operational Research 254 (2016) 875–888

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Stochastics and Statistics

An exact method for the sensitivity analysis of systems simulated by rejection techniques

Mark S. Joshi[a,*], Dan Zhu[b]

[a] Centre for Actuarial Studies, Department of Economics, University of Melbourne, Victoria 3010, Australia
[b] Department of Econometrics and Business Statistics, Monash University, Victoria 3145, Australia

ARTICLE INFO

Article history:
Received 13 May 2015
Accepted 13 April 2016
Available online 21 April 2016

Keywords:
Monte Carlo simulation
Sensitivity analysis
Acceptance–rejection sampling
Option pricing under Levy processes
Queuing theory

ABSTRACT

We compute first- and second-order sensitivities of functions simulated by rejection techniques. The methodology is to perform a measure change on every acceptance test, so that the pathwise discontinuities resulting from the rejection decisions are removed. The change of measure is chosen to be optimal in terms of minimizing variances of the likelihood ratio terms. Applications are presented for computing Greeks of equity options with certain Lévy-driven underlyings and to finding sensitivities of performance measures in queueing systems. The numerical results demonstrate the efficacy and speed of the method.

1. Introduction

Simulation is a widely-used tool in Operations Research. It can be viewed as a way to compute statistics of complex systems evolving under randomness or as a methodology for evaluating high-dimensional integrals. However, it is not enough just to compute values, for many applications the sensitivities of these values with respect to model inputs is equally important. When the parameters are estimated, knowing how inaccuracy in inputs affects the accuracy of outputs is crucial for decision making. When the parameters are known precisely but themselves random, sensitivities are an essential tool for risk assessment and designing risk mitigation strategies. This is particularly the case when using Monte Carlo simulation to price complex financial derivative products. For many OR applications, the objective is optimize some quantity as a function of the inputs, this is facilitated by good estimates of not just the gradient but also of the Hessian.

The simplest and most obvious approach to computing sensitivities is to bump the input parameters and see how the output changes. This method is generally called finite differencing (FD). It has the defects of bias and requiring one simulation per sensitivity. When the function, F, mapping the inputs to the outputs is discontinuous, FD yields very high variances rendering it unusable. When F is sufficiently smooth, analysis of the small bump limit yields the pathwise method, or infinitesimal perturbation analysis (IPA), which essentially consists of differentiating the value along each path as a function of the inputs. The pathwise method when applicable is generally very effective and low variance. (See eg Glasserman, 2004.) There has therefore been much effort devoted to addressing the problem of how to adapt it to cope with discontinuities in specific cases. For example, Hong and Liu (2011) write the objective function as an indicator function times a smooth function and then approximate the delta distribution that arises from differentiation by a Gaussian. Their approach results in bias, however, and the asymptotically unbiased method has convergence of order below 1/2. Glasserman (1992) suggests another approach that avoids differentiating discontinuities via multiplying by a function vanishing on them. Hong and Liu (2010) present an interesting approach to computing probabilities that an evolved quantity lies in a given set when the simulation algorithm is continuous.

A widely-used intrinsically-discontinuous simulation method is rejection sampling and there has been surprisingly little work on how to implement the pathwise method for it. Rejection techniques such as the acceptance–rejection method (Neumann, 1951), the ratio-of-uniforms method (Kinderman & Monahan, 1977) and the transformations-with-multiple-root method (Michael, Schucany, & Haas, 1976), are powerful alternatives for simulating random variates when the conventional inverse-transform method is not applicable. For example, various rejection methods are introduced to simulate the gamma and inverse-gaussian variates, which are essential for pricing exotic derivatives with the variance-gamma (VG) and the normal-inverse-gaussian (NIG) stock process. Furthermore, in the study of queuing systems, the rejection

* Corresponding author. Tel.: +61 383446899.
E-mail address: mark.joshi@gmail.com (M.S. Joshi).

http://dx.doi.org/10.1016/j.ejor.2016.04.024
0377-2217/

method “thinning” (Lewis & Shedler, 1979) is commonly used for simulating non-homogeneous poisson arrival processes. The fundamental issue for all these methods is that a small change in parameter values may alter the decision at each acceptance test and so render a very different outcome for a given sample. None of the adaptations of the pathwise method discussed above appear to be applicable to rejection sampling.

One alternative approach is to differentiate the underlying density and this yields the likelihood ratio method (LR). However, this requires easy computation of the density and also can lead to high variances so pathwise techniques are preferable when available. For many cases where rejection sampling is natural, the density is intractable. Another possible approach related to LR is that of measure-valued differentiation (MVD) which involves differentiating the underlying measure and so avoids simulation discontinuity problems. In particular, in recent work Volk-Makarewicz (2014) considers a similar problem for derivatives prices using the Variance Gamma process but does not compute sensitivities of the shape parameter for the gamma random variable. There does not seem to have been any work on the NIG process using MVD. Other related approaches are the vibrato technique of Giles (2009) and Malliavin techniques (Benhamou, 2003). Whilst all of these approaches are interesting, none of them are obviously adaptable to the case of rejection sampling.

An important aspect of pathwise methods is that the simulation algorithm is also differentiated. This implies a dependency on its choice. This can cause issues both in terms of applicability when the output of the algorithm is not continuous, and also of variance since the estimates depend on the derivatives of terms in the algorithm. Joshi and Zhu (2014) extended the Optimal Partial Proxy method (Chan & Joshi, 2015) to computing Hessians for financial products with discontinuous payoffs (HOPP). Here, we adapt the idea of Joshi and Zhu (2014) and introduce a change of measure function at each acceptance test, so the small bump in the parameter of interest does not alter the acceptance–rejection decision rendering the pathwise method applicable at the cost of a likelihood ratio term which we show is in a certain sense of minimal variance.

Practical problems in OR often involve multiple parameters of interest. This leads to another obstacle to calculating the Hessian, the associated computational effort. Efficient automatic simulation algorithms for computing Hessians have long been of interest to the community, we refer the readers to Griewank and Walter (2008) for an overview. Here, we adopt the Algorithmic Hessian approach introduced by Joshi and Yang (2011) for convenience. Their methodology is to decompose the evolution into elementary operations, then initialize the Hessian and gradient at the terminal time point and update them in a backward fashion with one elementary operation each step. We name the simulation algorithm resulting from combining our discontinuity removed rejection algorithm with the Joshi–Yang method, the Optimal Sensitivities for Rejection Sampling (OSRS) method.

We emphasize the widespread applicability of OSRS: it does not require the simulation algorithm nor the payoff function to be continuous; the marginal density is not explicitly needed; and the measure changes are benign and result in much smaller variances than LR when both are applicable. We present applications to both financial engineering and queuing systems to illustrate the method’s breadth. We present results for risk management of call and barrier options using VG and NIG processes: we are able to compute first- and second-order sensitivities to all parameters including the Gamma process’s shape parameter. In addition, we consider queueing systems that fail to satisfy the conditions in (Glasserman, 1991) for the application of IPA. We present sensitivities of such discrete-event systems with respect to both distributional and structural parameters.

The remaining sections of this paper are organized as follows. The basic idea of OSRS is presented in Section 2. In Section 3, we apply our OSRS to computing sensitivities of call options and barrier options with Lévy-driven underlyings. In Section 4, we apply the model to compute parameter sensitivities of the average time spent of a finite-time horizon $M_t|M|1$ queue, where the interarrival time is simulated by the thinning technique.

We thank the editor and some anonymous referees for their detailed comments on an earlier version of this paper.

2. The basic idea of OSRS

In this section, we provide a general framework for computing sensitivities of expectations of performance measures calculated via rejection techniques. Computing sensitivities is considered particularly hard when the underlying state variables can only be obtained via rejection sampling. The LR method is problematic since the underlying distributions for such cases do not have known and tractable probability density functions, and it has a tendency to produce high variances. The FD method produces biased estimates and it is not feasible for rejection sampling in that a small bump in the parameter of interest can cause a switch between acceptance and rejection, thus a significant change in the simulated outcome. The pathwise method as the limit case of the FD method cannot be applied here due to this inherent discontinuity of the algorithm.

2.1. Definitions and notations

Before presenting the basic idea of OSRS method, we first define a new class of functions.

Definition 1. Let $\hat{C}^2$ denote the class of functions that

- have Lipschitz continuous first-order derivatives everywhere,
- are twice differentiable almost surely.

In the following sections, we construct alternative $\hat{C}^2$ algorithms for computing performance measures, which consequently admit the application of the pathwise method to computing first- and second-order derivatives (Glasserman, 2004).

To fix the notations, let

- $\theta \in \mathbb{R}^m$ denote the parameters of interest within a small neighbourhood Θ about the base point θ_0,
- V_j denote the standard uniforms for generating the jth simulated outcome and $s(\theta, V_j)$ denote the algorithm for generating an outcome,
- V_j^D denote the jth decision standard-uniform variate,
- V denote the standard uniforms for generating a random variate from the target distribution including both V_j's and V_j^D's,
- $N(\theta, V)$ denote the number of simulated outcomes up to and including the accepted one,
- $S(\theta, V)$ denote the rejection algorithm for turning θ and V into the target random variable, so that

$$S(\theta, V) = s(\theta, V_{N(\theta,V)}). \tag{2.1}$$

The value $N(\theta, V) \in \mathbb{N}$ is itself a discrete random variable which depends on the parameter of interest. For example, in acceptance–rejection sampling, each outcome is generated from a related distribution with density function f^*, and accepted as a random variable from the target distribution with density function f, if an independent decision variate, V_j^D, satisfies,

$$V_j^D \le \frac{f\big(s(\theta, V_j)\big)}{c(\theta) f^*\big(s(\theta, V_j)\big)},$$

for $c(\theta)$ which depends on the distributional parameters. The probability of acceptance is $\frac{1}{c(\theta)}$, and the number of iterations needed follows a geometric distribution.