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On moment non-explosions for Wishart-based stochastic volatility models

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ABSTRACT

This paper provides a result on moment non-explosions for a stock following a Wishart multidimensional stochastic volatility dynamics or a Wishart affine stochastic correlation dynamics when the parameter values satisfy certain constraints. By reformulating the stock dynamics in terms of the volatility path along with standard results on matrix Lyapunov and Riccati equations, a non-explosion result of the moment of order greater than one can be obtained. It extends to these frameworks a property well known for the Heston model.

1. Introduction

In (Andersen & Piterbarg, 2007), the authors consider the problem of moment explosions for different stochastic volatility models. They show that under certain constraints on the parameters the moment of order higher than one of the stock can explode in finite time. By complementarity, for certain parameter values there is no explosion, a result that is also useful for option pricing using a Fourier transform approach. This kind of result is quite easy to obtain for the model proposed by Heston (1993) as the moment-generating function is known in closed form.

The Wishart Multidimensional Stochastic Volatility model (WMSV), proposed in (Da Fonseca, Grasselli, & Tebaldi, 2008), and the Wishart Affine Stochastic Correlation model (WASC), proposed in (Da Fonseca, Grasselli, & Tebaldi, 2007), are extensions of the Heston model whose main feature is that they involve for the volatility dynamics a Wishart process proposed by Bru (1991). The WMSV model is a single-stock multidimensional stochastic volatility model while the WASC model is a multi-asset stochastic volatilityn-explosions directly from the moment-generating function of the stock is not feasible (or at least we did not manage to obtain such kind of results). However, by rewriting the stock price dynamics as a function of the volatility path and by performing a convenient conditioning, similar to the one used in (Da Fonseca, Gnoatto, & Grasselli, 2015), the problem can be reformulated in terms of the Wishart moment-generating function for which standard results allow us to draw conclusions on moment non-explosions in these two models.

The structure of the paper is as follows: in Section 2 we present the models and the moment-generating functions; in Section 3 we provide the results on moment non-explosions; in Section 4 numerical examples based on real parameter values are given; the last section concludes.

2. The models

2.1. The WMSV model

In this section we briefly review the main results regarding the WMSV proposed in (Da Fonseca et al., 2008). In this model the volatility is described by the Wishart process, a matrix-valued stochastic process defined in (Bru, 1991) and introduced in finance in (Gouriéroux & Sufana, 2010), and the dynamics for the stock price which is given by the following stochastic differential equation (SDE):

$$ds_t = s_t \mathrm{Tr}\left[\sqrt{\Sigma_t}\left(dW_t R^\top + dB_t\sqrt{\mathbb{I}_n - RR^\top}\right)\right], \qquad (1)$$

with $s_0 > 0$, Tr is the trace operator, $\mathbb{I}_n$ is the identity matrix of size n, $W_t, B_t \in M_n$ (the set of square matrices) are composed by n^2 independent Brownian motions under the risk-neutral measure (B_t and W_t are independent), $R \in M_n$ represents the correlation matrix and Σ_t belongs to S_n^+, the set of $n \times n$ symmetric positive semi-definite matrices (we suppose that the risk free rate is zero). In this specification the volatility is multi-dimensional and depends

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on the elements of the matrix process Σ_t, which is assumed to satisfy the following dynamics:

$$d\Sigma_t = \left(\Omega^2 + M\Sigma_t + \Sigma_t M^\top\right)dt + \sqrt{\Sigma_t}dW_t Q + Q dW_t^\top \sqrt{\Sigma_t}, \quad (2)$$

with $\sqrt{\cdot}$ denoting the matrix square root, initial condition Σ_0 a strictly positive definite matrix, $M \in M_n$, $\Omega \in S_n^{++}$ and $Q \in S_n^{++}$, the set of $n \times n$ symmetric definite positive matrices.

Eq. (2) characterizes the Wishart process and represents the matrix analogue of the square root mean-reverting process. In order to grant the typical mean reverting feature of the volatility, the matrix M is assumed to be such that $\lambda_i^m < 0; i = 1 \dots n$ with $\lambda_i^m \in \text{Spec}(M)$ (i.e. the spectra of M). The constant drift part satisfies $\Omega^2 = \beta Q^2$ with the real parameter $\beta \geq n-1$ and ensures that Σ_t is a (symmetric) positive semi-definite matrix. If β satisfies the stronger assumption $\beta \geq n+1$ then the unique strong solution to the SDE (2) evolves in S_n^{++}, see Mayerhofer, Pfaffel, and Stelzer (2011). The constraint on Ω can be relaxed as explained in (Cuchiero, Filipovic, Mayerhofer, & Teichmann, 2011) but we keep this more parsimonious choice in view of numerical applications as to the best of our knowledge it is the only specification for which a calibration on market option prices is available. This model and the other models presented in this work belong to the class of affine processes, see for example Duffie, Filipovic, and Schachermayer (2003) and Keller-Ressel and Mayerhofer (2015).

Remark 2.1. In (Da Fonseca et al., 2015) the noise of (2) is given by $\sqrt{\Sigma_t}dW_t Q + Q^\top dW_t^\top \sqrt{\Sigma_t}$ with $Q \in GL(n)$, the set of invertible $n \times n$ matrices. The polar decomposition of $Q = U\tilde{Q}$ with U a unitary matrix and $\tilde{Q} \in S_n^{++}$ along with the invariance of the law of the Brownian motion W_t with respect to unitary transformations imply that the dynamics proposed here is similar to the one considered in (Da Fonseca et al., 2015). Lastly, $dW_t R^\top + dB_t\sqrt{\mathbb{I}_n - RR^\top} = dW_t U (RU)^\top + dB_t \sqrt{\mathbb{I}_n - (RU)(RU)^\top}$ and $\tilde{R} = RU$ is the correlation matrix.

In this model the instantaneous variance of the asset returns is associated to the trace of the Wishart matrix, that is: $d\langle \ln s_t\rangle = \text{Tr}[\Sigma_t]dt$, which alone is not Markovian and constitutes also a multivariate extension of the Heston model.

Lemma 2.1. *(see Da Fonseca et al. (2008)) Given a scalar z and two square (symmetric) matrices $\Lambda_\Sigma, \Lambda_I$, the joint moment generating function of $(\ln s_t, \Sigma_t, \int_0^t \Sigma_u du)$ is denoted $G_{\text{WMSV}}(t, z, \Lambda_\Sigma, \Lambda_I)$ and given by*

$$\mathbb{E}[e^{z\ln s_t + \text{Tr}[\Lambda_\Sigma \Sigma_t] + \text{Tr}[\Lambda_I \int_0^t \Sigma_u du]}] = e^{z \ln s_0 + \text{Tr}[A(t)\Sigma_0] + b(t)}, \quad (3)$$

where the deterministic matrix function $A(t)$ and the scalar function $b(t)$ satisfy the following ODE (ordinary differential equations) where we omit the time variable t:

$$\frac{dA}{dt} = A\left(M + zQR^\top\right) + \left(M + zQR^\top\right)^\top A + 2AQ^2A + \frac{z(z-1)}{2}\mathbb{I}_n + \Lambda_I, \quad (4)$$

$$\frac{db}{dt} = \text{Tr}[\Omega^2 A], \quad (5)$$

with initial conditions $A(0) = \Lambda_\Sigma$, $b(0) = 0$. The solution is explicitly given by:

$$A(t) = (\Lambda_\Sigma A_{12}(t) + A_{22}(t))^{-1}(\Lambda_\Sigma A_{11}(t) + A_{21}(t)), \quad (6)$$

$$b(t) = -\frac{\beta}{2}\text{Tr}\left[\log(\Lambda_\Sigma A_{12}(t) + A_{22}(t)) + t(M + zQ^\top R^\top)\right], \quad (7)$$

with

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} = \exp t\mathcal{A} \quad (8)$$

and

$$\mathcal{A} = \begin{pmatrix} M + zQR^\top & -2Q^2 \\ \frac{z(z-1)}{2}\mathbb{I}_n + \Lambda_I & -\left(M + zQR^\top\right)^\top \end{pmatrix}.$$

When $n = 1$ then the model corresponds to the Heston model presented in (Heston, 1993). In that case the dynamics can be written as

$$ds_t = s_t\sqrt{v_t}(\rho dw_{1,t} + \sqrt{1-\rho^2}dw_{2,t}), \quad (9)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dw_{1,t}, \quad (10)$$

with $s_0 > 0$, $v_0 > 0$, $w_t = (w_{1,t}, w_{2,t})_{t \geq 0}$ a two-dimensional Brownian motion, $\kappa \in \mathbb{R}$, $\kappa\theta \in \mathbb{R}_+$, $\sigma > 0$ and $\rho \in [-1, 1]$.

The moment-generating function for the (log) stock s_t is known in closed form as we have the following lemma.

Lemma 2.2. *In the model proposed by Heston (1993), the moment-generating function of $(\ln s_t)$, defined by $G_{\text{HES}}(t, z) = \mathbb{E}\left[e^{z \ln s_t}\right]$, is equal to $\exp(z \ln s_0 + A(t)v_0 + b(t))$ with the deterministic functions $A(t)$, $b(t)$ defined as:*

$$A(t) = \frac{z^2 - z}{2}\frac{1 - e^{-\sqrt{\Gamma}t}}{\lambda_+ - \lambda_- e^{-\sqrt{\Gamma}t}}, \quad (11)$$

$$b(t) = \frac{2\kappa\theta}{\sigma^2}\left(t\lambda_- - \log\left(\frac{\lambda_+ - \lambda_- e^{-\sqrt{\Gamma}t}}{\lambda_+ - \lambda_-}\right)\right), \quad (12)$$

with $2\lambda_\pm = (\kappa - z\rho\sigma) \pm \sqrt{\Gamma}$, $\Gamma = (\kappa - z\rho\sigma)^2 - \sigma^2(z^2 - z)$.

We remind the reader of the main result on moment non-explosion of Andersen and Piterbarg (2007). If $z > 1$ but still such that $\Gamma > 0$ we deduce that $\sqrt{\Gamma} < |\kappa - \rho\sigma z|$. If $\rho < 0$ (that will be the case in practice) then $\lambda_- > 0$ (and still $\lambda_+ > 0$) so t^* such that $\lambda_+ - \lambda_- e^{-\sqrt{\Gamma}t^*} = 0$ is equivalent to $t^* = -\frac{1}{\sqrt{\Gamma}}\ln(\frac{\lambda_+}{\lambda_-})$. As $\lambda_+ - \lambda_- = \sqrt{\Gamma} > 0$ we conclude that $t^* < 0$, hence there isn't a moment explosion. The fact that $\rho < 0$ is the favourable case is intuitive as an increase of the stock implies a decrease of the volatility and therefore a thinner upper tail for the stock.

Remark 2.2. The moment explosions problem has attracted quite a lot of attention among academics during the past few years. In addition to the work aforementioned let us also mention the works of Lions and Musiela (2007) and Keller-Ressel (2011).

As discussed in (Da Fonseca et al., 2015), in the WMSV model the stock's variance, the variance of stock's variance and the correlation between the log-stock and its (instantaneous) variance are given by (in the particular case of $n = 2$)

$$\begin{aligned} d\langle \ln s_t\rangle &= \text{Tr}[\Sigma_t]dt \\ d\langle \text{VAR}(s_t)\rangle &= Q_{11}^2\Sigma_t^{11} + Q_{22}^2\Sigma_t^{22}dt \\ d\text{CORR}(\ln s_t, \text{VAR}(s_t)) &= \frac{\text{Tr}[RQ\Sigma_t]}{\sqrt{\text{Tr}[\Sigma_t]}\sqrt{\text{Tr}[Q^\top Q\Sigma_t]}}dt. \end{aligned} \quad (13)$$

The correlation between the log-stock and its (instantaneous) variance, given by (13), is found to be negative in practice (see Da Fonseca & Grasselli (2011)); that is why it was conjectured in (Da Fonseca et al., 2015) that there should be $z > 1$ such that $\mathbb{E}\left[s_t^z\right] = G_{\text{WMSV}}(t, z, 0_n, 0_n) < \infty \; \forall t > 0$. Hence, a moment greater than one exits for which no explosion occurs. This aspect is also important from an implementation point of view as to perform the pricing of a call option using the Fourier transform, as proposed by Carr and Madan (1999),