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Decision Support

Optimal weights in DEA models with weight restrictions

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ABSTRACT

According to a conventional interpretation of a multiplier DEA model, its optimal weights show the decision making unit under the assessment, denoted DMU_o, in the best light in comparison to all observed DMUs. For multiplier models with additional weight restrictions such an interpretation is known to be generally incorrect (specifically, if weight restrictions are linked or nonhomogeneous), and the meaning of optimal weights in such models has remained unclear. In this paper we prove that, for any weight restrictions, the optimal weights of the multiplier model show DMU_o in the best light in comparison to the entire technology expanded by the weight restrictions. This result is consistent with the fact that the dual envelopment DEA model benchmarks DMU_o against all DMUs in the technology, and not only against the observed DMUs. Our development overcomes previous concerns about the use of weight restrictions of certain types in DEA models and provides their rigorous and meaningful interpretation.

1. Introduction

Data envelopment analysis (DEA) is a nonparametric approach to the assessment of efficiency and productivity of organizational units (Cooper, Seiford, & Tone, 2007; Thanassoulis, Portela, & Despić, 2008). The latter are conventionally referred to as decision making units (DMUs). Standard DEA models are based on the assumption that the underlying *production technology* is characterized by either constant (CRS) or variable (VRS) returns to scale.

Both CRS and VRS models can be stated as two mutually dual linear programs referred to as the *envelopment* and *multiplier* models. The optimal value of these two programs is interpreted as the input or output radial efficiency of DMU_o under the assessment, depending on the orientation in which the models are solved (Banker, Charnes, & Cooper, 1984; Charnes, Cooper, & Rhodes, 1978). In particular, in the envelopment model, DMU_o is benchmarked against the boundary of the CRS or VRS technology, and the radial efficiency of DMU_o is interpreted as the utmost proportional improvement factor to its input or output vector possible in the technology.

The multiplier models are stated in terms of variable input and output weights (multipliers). The CRS multiplier model can be shown to maximize the ratio of the total weighted output to the total weighted input (*efficiency ratio*) of DMU_o, provided no such ratio across all observed DMUs can exceed the value of 1. The VRS multiplier model has an additional dual variable interpretable in terms of returns to scale and scale elasticity (Banker et al., 1984; Podinovski, Chambers, Atici, & Deineko, 2016; Podinovski & Førsund, 2010; Podinovski, Førsund, & Krivonozhko, 2009; Sahoo & Tone, 2015). As pointed by Charnes et al. (1978), the optimal input and output weights are the most favorable to DMU_o and show it in the best light in comparison to all observed DMUs.

1.1. Weight restrictions

Weight restrictions usually represent value judgments incorporated in the form of additional constraints on the input and output weights in the multiplier model. These constraints reduce the flexibility of weights and typically improve the discrimination of the DEA model (see, e.g., Allen, Athanassopoulos, Dyson, & Thanassoulis, 1997; Cook & Zhu, 2008; Joro & Korhonen, 2015; Thanassoulis et al., 2008).

The use of weight restrictions generally changes the interpretation of efficiency in both the envelopment and multiplier models. From the technology perspective, the incorporation of weight restrictions results in the expansion of the model of technology (Charnes, Cooper, Wei, & Huang, 1989; Halme & Korhonen, 2000; Roll, Cook, & Golany, 1991). Podinovski (2004a) shows that this expansion is caused by the dual terms in the envelopment model generated by weight restrictions, and that DMU_o is projected on the boundary of the expanded technology. Therefore, DMU_o is benchmarked against all units in the technology (including those generated by the weight restrictions), and *not only* against the observed units.

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The interpretation of efficiency in terms of the multiplier model with weight restrictions is somewhat less obvious and currently incomplete. This can be summarized as follows. If all weight restrictions are homogeneous and not linked (see Section 2 for a formal definition), the multiplier model correctly identifies the optimal weights (within the specified weight restrictions) that represent DMU_o in the best light in comparison to all observed DMUs (Podinovski, 2001a).

However, a problem with the interpretation arises if at least one weight restriction is nonhomogeneous or is linked. In this case the optimal weights do not generally represent DMU_o in the best light in comparison to all observed DMUs. Consequently, the optimal value of the multiplier model with such weight restriction generally underestimates the relative efficiency of DMU_o. Examples illustrating this point are given by Podinovski (1999, 2001a); Podinovski and Athanassopoulos (1998) and, recently, by Khalili, Camanho, Portela, and Alirezaee (2010).

1.2. Contribution

In this paper we show that, for *any* weight restrictions, the optimal weights of the multiplier model show DMU_o in the best light in comparison to all DMUs in the *expanded technology* generated by the weight restrictions. This result is true if we search among all nonnegative input and output weights, or only among those that satisfy the weight restrictions.

Our results also overcome the discrepancy between the interpretation of the envelopment and multiplier models with weight restrictions. Indeed, as pointed above, the envelopment model benchmarks DMU_o against *all* DMUs in the technology expanded by the weight restrictions. However, the conventional interpretation of the multiplier model assumes that DMU_o should be benchmarked against the observed DMUs only. As noted, this conventional assumption does not lead to a meaningful interpretation of some types of weight restrictions. Our results show that the multiplier model does *exactly the same* as the envelopment model—it benchmarks DMU_o against *all* DMUs in the expanded technology, for all types of weight restrictions.

From a practical perspective, this new interpretation can be used to justify the incorporation of any types of weight restrictions in the multiplier model, and explain the meaning of the resulting optimal weights and efficiency scores. This includes absolute weight bounds and linked weight restrictions, whose meaning has so far remained unclear.

2. Weight restrictions and production trade-offs

To be specific, we derive our main results for the input-oriented models under the assumption of CRS. These results fully extend to the output-oriented models and also to the case of VRS, with obvious minor modifications as outlined in Section 5.

2.1. Multiplier models with weight restrictions

Consider the set of observed DMUs (X_j, Y_j), $j = 1, \ldots, N$, where $X_j \in \mathbb{R}^m_+ \setminus \{0\}$ and $Y_j \in \mathbb{R}^s_+ \setminus \{0\}$ are, respectively, the vectors of inputs and outputs. The DMU_o under the assessment is denoted (X_o, Y_o).

Multiplier CRS models are stated in terms of variable vectors of input and output weights $v \in \mathbb{R}^m_+$ and $u \in \mathbb{R}^s_+$. Weight restrictions are additional constraints on vectors v and u incorporated in the multiplier model and stated in the general form as follows:

$$Q_t^\top u - P_t^\top v \le c_t, \quad t = 1, \ldots, K. \tag{1}$$

In inequalities (1), components of vectors $Q_t \in \mathbb{R}^s$ and $P_t \in \mathbb{R}^m$, and constant scalars c_t may be positive, negative or zero. The weight restriction t is *linked* if both vectors P_t and Q_t are nonzero, and *not linked* otherwise. The weight restriction t is *homogeneous* if $c_t = 0$, and *nonhomogeneous* otherwise.[1,2]

Remark 1. Using the normalizing equality of the multiplier models, any nonhomogeneous weight restriction can be replaced by a homogeneous one. For example, using equality (2.2) stated below, a nonhomogeneous weight restriction t is replaced by the homogeneous (possibly linked) weight restriction which, after a simple rearrangement, takes on the form $Q_t^\top u - (P_t + c_t X_o)^\top v \le 0$.[3]

Based on Remark 1 and therefore *without loss of generality* we assume that all weight restrictions (1) are homogeneous. The input-oriented CRS multiplier model with such weight restrictions is stated as follows:

$$\theta^* = \max \quad Y_o^\top u \tag{2.1}$$

$$\text{subject to} \quad X_o^\top v = 1, \tag{2.2}$$

$$Y_j^\top u - X_j^\top v \le 0, \quad j = 1, \ldots, N, \tag{2.3}$$

$$Q_t^\top u - P_t^\top v \le 0, \quad t = 1, \ldots, K, \tag{2.4}$$

$$u, v \ge 0. \tag{2.5}$$

2.2. Envelopment models with production trade-offs

To demonstrate that weight restrictions (2.4) result in the expansion of the standard CRS technology, consider the dual envelopment model to program (2):

$$\theta^* = \min \quad \theta \tag{3.1}$$

$$\text{subject to} \quad \sum_{j=1}^N \lambda_j X_j + \sum_{t=1}^K \pi_t P_t + S_X = \theta X_o, \tag{3.2}$$

$$\sum_{j=1}^N \lambda_j Y_j + \sum_{t=1}^K \pi_t Q_t - S_Y = Y_o, \tag{3.3}$$

$$\lambda, \pi, S_X, S_Y \ge 0, \theta \text{ sign free}. \tag{3.4}$$

The above model allows a straightforward interpretation. The DMU

$$(\hat{X}, \hat{Y}) = \left(\sum_{j=1}^N \lambda_j X_j, \sum_{j=1}^N \lambda_j Y_j \right)$$

in equalities (3.2) and (3.3) is a unit in the standard CRS technology. DMU $(\hat{X}, \hat{Y})$ is further modified by the terms generated by weight restrictions (2.4):

$$(P_t, Q_t), \quad t = 1, \ldots, K. \tag{4}$$

[1] Following Charnes et al. (1989), unlinked homogeneous weight restrictions are often referred to as assurance regions of Type I. A special case of this type is virtual weight restrictions of Wong and Beasley (1990). Similarly, following Thompson, Langemeier, Lee, Lee, and Thrall (1990), linked homogeneous weight restrictions are referred to as assurance regions of Type II. The most common example of nonhomogeneous weight restrictions is absolute weight bounds (Dyson & Thanassoulis, 1988).

[2] DEA literature suggests different methods for assessing weight restrictions of various types (see, e.g., reviews in Thanassoulis et al., 2008 and Jain, Kumar, Kumar, & Chandra, 2015). Our new results apply to any weight restrictions (1), regardless of the method used for their assessment.

[3] The described transformation obviously depends on the DMU_o under the assessment and also on the (input or output) orientation of the model (Podinovski, 2004a, 2005).