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Interfaces with Other Disciplines

On calibration of stochastic and fractional stochastic volatility models

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ABSTRACT

In this paper we study optimization techniques for calibration of stochastic volatility models to real market data. Several optimization techniques are compared and used in order to solve the nonlinear least squares problem arising in the minimization of the difference between the observed market prices and the model prices. To compare several approaches we use a popular stochastic volatility model firstly introduced by Heston (1993) and a more complex model with jumps in the underlying and approximative fractional volatility. Calibration procedures are performed on two main data sets that involve traded DAX index options. We show how well both models can be fitted to a given option price surface. The routines alongside models are also compared in terms of out-of-sample errors. For the calibration tasks without having a good knowledge of the market (e.g. a suitable initial model parameters) we suggest an approach of combining local and global optimizers. This way we are able to retrieve superior error measures for all considered tasks and models.

1. Introduction

In finance, stochastic volatility (SV) models are used to evaluate derivative securities, such as options. These models were developed out of a need to modify the Nobel price winning (Black & Scholes, 1973) model for option pricing, which failed to effectively take the volatility in the price of the underlying security into account. The Black Scholes model assumed that the volatility of the underlying security was constant, while SV models consider it to be a stochastic process. Among the first publications about stochastic volatility models were Hull and White (1987), Scott (1987), Stein and Stein (1991) and Heston (1993).

Later several extensions to SV models were proposed. In particular, to fit the short term prices, a model with stochastic volatility and jumps was introduced by Bates (1996), who combined approaches of Heston (1993) and Merton (1976). Furthermore, in order to capture volatility clustering phenomenon in the SV model explicitly, long memory driving process in volatility was used for example by Intarasit and Sattayatham (2011). This property is described by a long memory parameter named after hydrologist H. E. Hurst. Its value can be estimated from the realized volatility time-series as in Bollerslev and Mikkelsen (1996), Breidt, Crato, and de Lima (1998) and Martens, van Dijk, and de Pooter (2004), or it can be obtained from the calibration to the market data.

Calibration is the process of identifying the set of model parameters that are most likely given by the observed data. Heston model was the first model that allowed reasonable calibration to the market option data together with semi-closed form solution for European call/put option prices. Heston model also allows correlation between the asset price and the volatility process as opposed to Stein and Stein (1991). Although the Heston model was already introduced in 1993 and several other SV models appeared, it is nowadays still one of the most popular models for option pricing.

Many other SV models have been introduced since, including a more flexible version of the Heston model which involves time-dependent parameters. The case of piece-wise constant parameters in time is studied in Nögel and Mikhailov (2003), a linear time dependence in Elices (2008) and a more general case is introduced in Benhamou, Gobet, and Miri (2010). The later result involves only an approximation to the option price. However, Bayer, Friz, and Gatheral (2015) suggest that the general overall shape of the volatility surface does not change in time, at least to a first approximation. Hence, it is desirable to model volatility by a time-homogeneous process. Other generalizations of the Heston model with time-constant parameters include jump processes in asset price, in volatility or in both (see e.g Duffie, Pan, & Singleton, 2000).

The industry standard approach to calibration is to minimize the difference between the observed prices and the model prices. Option pricing models are calibrated to prices observed on the market in order to compute over-the-counter derivative prices or

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hedge ratios. The complexity of the model calibration process increases with more realistic models and the fact that the estimation method of model parameters becomes as crucial as the model itself is mentioned by Jacquier and Jarrow (2000).

In our case, the input parameters cannot be directly observed from the market data, thus empirical estimates are of no use. It was well documented in Bakshi, Cao, and Chen (1997) that the model implied parameters differ significantly from their time-series estimated counterparts. For instance, the magnitudes of time-series correlation coefficient of the asset returns and its volatility estimated from the daily prices were much lower than their model implied counterparts.

Moreover, the information observed from market data is insufficient to exactly identify the parameters, because several sets of parameters may be performing well and provide us with model prices that are close to the prices observed on the market. This is what causes the ill-posedness of the calibration problem.

The paper is organized as follows. In Section 2 we briefly introduce the stochastic volatility models under consideration, in particular the Heston model and the approximative fractional model together with their semi-closed form solutions for vanilla options. In Section 3 we introduce the testing methodology – most importantly we disclose how we measure the model performance, how calibration tasks are formulated and we also comment in detail on the data structure. Among the considered methods there are three global optimizers, i.e. genetic algorithm (GA), simulated annealing (SA) and adaptive simulated annealing (ASA) as well as the local search method (denoted by LSQ).

In Section 4 we demonstrate how the optimization procedures can be used for the calibration problem on particular data sets. We will conclude our results in Section 5.

2. Stochastic volatility models

2.1. Heston model

Following Heston (1993) and Rouah (2013) we consider the risk-neutral stock price model:

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\widetilde{W}_t^S, \tag{1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} d\widetilde{W}_t^v, \tag{2}$$

$$d\widetilde{W}_t^S d\widetilde{W}_t^v = \rho\, dt, \tag{3}$$

with initial conditions $S_0 \geq 0$ and $v_0 \geq 0$, where S_t is the price of the underlying asset at time t, v_t is the instantaneous variance at time t, r is the risk-free rate, θ is the long run average price variance, κ is the rate at which v_t reverts to θ and σ is the volatility of the volatility. $(\widetilde{W}^S, \widetilde{W}^v)$ is a two-dimensional Wiener process under the risk-neutral measure $\widetilde{\mathbb{P}}$ with instantaneous correlation ρ.

Stochastic process v_t is referred to as the variance process (also known as volatility process) and it is the square-root mean reverting process, CIR process (Cox, Ingersoll, & Ross, 1985). It is strictly positive and cannot reach zero if the Feller condition $2\kappa\theta > \sigma^2$ is satisfied (Feller, 1951).

Heston SV model allows for a semi-closed form solution for vanilla option, which involves numerical computation of an integral. Several pricing formulas were added to the original one by Heston (1993) in order to overcome numerical problems that the integrand poses. The following formulation by Albrecher, Mayer, Schoutens, and Tistaert (2007) eliminates the possible discontinuities in the integrand by only simple modifications of the original formula by Heston. Let K be the strike price and $\tau = T - t$ be the time to maturity. Then the price of a European call option at time t on a non-dividend paying stock with a spot price S_t is

$$V(S, v, \tau) = SP_1 - e^{-r\tau} KP_2, \tag{4}$$

$$P_j(x, v, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln(K)} f_j(x, v, \tau, \phi)}{i\phi}\right] d\phi,$$

where $x = \ln S$ and

$$f_j(x, v, \tau, \phi) = \exp\{C_j(\tau, \phi) + D_j(\tau, \phi)v + i\phi x\},$$

and where

$$C_j(\tau, \phi) = r\phi i\tau + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\phi i - d)\tau - 2\ln\left[\frac{1 - ge^{-d\tau}}{1 - g}\right]\right\},$$

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma\phi i - d}{\sigma^2}\left[\frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}\right],$$

$$g = \frac{b_j - \rho\sigma\phi i - d}{b_j - \rho\sigma\phi i + d},$$

$$d = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)},$$

for both $j = 1, 2$, where the parameters u_j, a and b_j are defined as follows:

$$u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = \kappa\theta, b_1 = \kappa - \rho\sigma, b_2 = \kappa.$$

Different approaches are taken in e.g. Kahl and Jäckel (2005), Lewis (2000) or Zhylyevskyy (2012). We will use here the formula by Lewis (2000), which is well-behaved and compared to the formulation by Albrecher et al. (2007) requires the numerical computation of only one integral for each call option price.

$$V(S, v, \tau) = S - Ke^{-r\tau}\frac{1}{\pi}\int_{0+i/2}^{\infty+i/2} e^{-ikX}\frac{\hat{F}(k, v, \tau)}{k^2 - ik} dk, \tag{5}$$

where $X = \ln(S/K) + r\tau$ and

$$\hat{F}(k, v, \tau) = \exp\left(\frac{2\kappa\theta}{\sigma^2}\left[qg - \ln\left(\frac{1 - he^{-\xi q}}{1 - h}\right)\right] + vg\left(\frac{1 - e^{-\xi q}}{1 - he^{-\xi q}}\right)\right),$$

where

$$g = \frac{b - \xi}{2}, \quad h = \frac{b - \xi}{b + \xi}, \quad q = \frac{\sigma^2\tau}{2},$$

$$\xi = \sqrt{b^2 + \frac{4(k^2 - ik)}{\sigma^2}},$$

$$b = \frac{2}{\sigma^2}\left(ik\rho\sigma + \kappa\right).$$

The Lewis formula (5) uses the (inverse) complex Fourier transform of the so called fundamental transform $\hat{F}(k, v, \tau)$, where k is complex-valued. Given the fundamental transform (of the corresponding pricing partial differential equation) one can obtain an option price for different particular payoff functions, not only the European call. Equivalence of the Lewis and Heston (and hence Albrecher) formulas can be found for example in Baustian, Mrázek, Pospíšil, and Sobotka (2016).

2.2. Model with approximative fractional stochastic volatility

We also consider a model with approximative fractional stochastic volatility that was motivated by Intarasit and Sattayatham (2011) and firstly introduced by Pospíšil and Sobotka (2015). Under a risk-neutral measure, the model dynamics takes the following form:

$$dS_t = (r - \lambda\beta)S_t\, dt + \sqrt{v_t} S_t\, d\widetilde{W}_t^S + S_{t-}\, dQ_t, \tag{6}$$

$$dv_t = \kappa(\theta - v_t)\, dt + \sigma\sqrt{v_t}\, d\widetilde{B}_t^{\varepsilon,H}, \tag{7}$$