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Discrete Optimization

Generalized linear fractional programming under interval uncertainty

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ABSTRACT

Data in many real-life engineering and economical problems suffer from inexactness. Herein we assume that we are given some intervals in which the data can simultaneously and independently perturb. We consider a generalized linear fractional programming problem with interval data and present an efficient method for computing the range of optimal values. The method reduces the problem to solving from two to four real-valued generalized linear fractional programs, which can be computed in polynomial time using an appropriate interior point method solver.

We consider also the inverse problem: How much can data of a real generalized linear fractional program vary such that the optimal values do not exceed some prescribed bounds. We propose a method for calculating (often the largest possible) ranges of admissible variations; it needs to solve only two real-valued generalized linear fractional programs. We illustrate the approach on a simple von Neumann economic growth model.

1. Introduction

Uncertainties in data measurement and observation is a common phenomenon in practice. Considering their interval envelopes is one way to tackle these uncertainties. Computing with interval values has many useful properties, e.g., it ensures that all possible instances of interval data are taken into account. Contrary to the traditional sensitivity analysis, this approach can handle simultaneous and independent perturbations of selected parameters.

Mathematical programming problems with interval data have been investigated for several decades. Many papers studied the problem of computing the range of optimal values of linear programming problem with data varying inside intervals, see [5,7,14,21] among others. Less people were involved nonlinear programming with data perturbing inside intervals. For instance, interval convex quadratic programming was studied in [13,20], posynomial geometric programming in [13,17–19], and a specific nonlinear programming problem with linear constraints in [29].

In this paper, we focus on a generalized linear fractional programming problem the data of which vary inside some given intervals. To the best of our knowledge, this problem itself has never been investigated. In the essence, it can be solved by the general method from [13], where a unified method for dealing with interval nonlinear programming problems was proposed. That approach was based on duality theory in nonlinear programming, and for generalized linear fractional programming we have a developed duality [6,15] to use. Nevertheless, the approach is a bit cumbersome: We have to derive characterization of primal and dual interval solution sets and the results will be restricted by some assumptions. Stronger results with no needless assumptions are obtained by direct inspection, which is exactly what we do in Section 2.

In Section 2, we show that the exact range of optimal values can be calculated by solving up to four real-valued mathematical programs. Moreover, the method is easily adapted for solving the inverse problem (Section 3): We are given real-valued a generalized linear fractional programming problem and some bounds on the optimal value function, and we calculate tolerances (intervals) for all required parameters such that the optimal values do not exceed the bounds while the parameters are perturbing inside their intervals.

Many applications of generalized linear fractional programming arise in the field of economics and optimization. For instance, von Neumann growth model of expanding economy [25], goal programming with rational criteria [3,4], or Chebyshev discrete rational approximation. For another applications, see e.g. [23,24]. Since the economical parameters of the real life problems (including the mentioned applications) are often imprecise, the results developed in this paper form a useful and efficient tool in decision making and analysis.

2. Range of optimal values

Consider a generalized linear fractional programming problem

$$f(A,B,C,c) := \inf \lambda \text{ subject to } Ax \leqslant \lambda Bx,\ Cx \leqslant c,\ x \geqslant 0, \qquad (1)$$

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where $A,\ B \in \mathbb{R}^{m\times n},\ C \in \mathbb{R}^{l\times n}$ and $c \in \mathbb{R}^l$. Moreover, we assume that $Bx \geqslant 0$ holds for all x satisfying $Cx \leqslant c,\ x \geqslant 0$. Such problems are solvable in polynomial time using an interior point method [8,22].

Now suppose that the input data are not known exactly, and we are given only lower and upper bounds on their values. Formally, the matrix A varies in some interval matrix $\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} | \underline{a}_{ij} \leqslant a_{ij} \leqslant \bar{a}_{ij}\}$, where $\underline{A},\ \overline{A} \in \mathbb{R}^{m\times n}$ are given matrices. In a similar way we consider interval matrices $\mathbf{B}$ and $\mathbf{C}$ and interval vector $\mathbf{c}$ in which B, C and c may perturb, respectively. Thus we have a family of problems (1) with $A \in \mathbf{A},\ B \in \mathbf{B},\ C \in \mathbf{C}$ and $c \in \mathbf{c}$. Any problem belonging to this family is referred as *an instance*.

To ensure that each instance is a proper generalized linear fractional programming problem we have to assume:

(A1) For every $B \in \mathbf{B},\ C \in \mathbf{C}$ and $c \in \mathbf{c}$ any solution to $Cx \geqslant c,\ x \geqslant 0$ solves also $Bx \geqslant 0$.

Proposition 1 shows that to verify this assumption; it suffices to verify only one instance with $B = \underline{B},\ C = \underline{C}$ and $c = \bar{c}$.

Proposition 1. *Assumption (A1) is true if and only if $\underline{B}x \geqslant 0$ holds for all x satisfying $\underline{C}x \leqslant \bar{c},\ x \geqslant 0$.*

Proof. One implication is easily seen as $B = \underline{B},\ C = \underline{C}$ and $c = \bar{c}$ is an instance of our family of problems.

Conversely, let $B \in \mathbf{B},\ C \in \mathbf{C}$ and $c \in \mathbf{c}$ and suppose that any x satisfying $\underline{C}x \leqslant \bar{c},\ x \geqslant 0$ is also a solution of $\underline{B}x \geqslant 0$. Now, let x^* be any solution to $Cx \leqslant c,\ x \geqslant 0$. Then

$$\underline{C}x^* \leqslant Cx^* \leqslant c \leqslant \bar{c}.$$

Thus x^* is a solution to $\underline{C}x \leqslant \bar{c}$, and by our supposition x^* solves also $\underline{B}x \geqslant 0$. Hence

$$Bx^* \geqslant \underline{B}x^* \geqslant 0.$$

Therefore x^* is a solution to $Bx \geqslant 0$. □

As data are perturbing in their intervals, the optimal value $f(A,B,C,c)$ ranges in some interval as well. Our aim is to determine the exact lower and upper bound on the optimal value. They are respectively defined as

$$\underline{f} := \inf f(A,B,C,c) \text{ subject to } A \in \mathbf{A},\ B \in \mathbf{B},\ C \in \mathbf{C},\ c \in \mathbf{c},$$

$$\bar{f} := \sup f(A,B,C,c) \text{ subject to } A \in \mathbf{A},\ B \in \mathbf{B},\ C \in \mathbf{C},\ c \in \mathbf{c}.$$

The following theorem says that both bounds can be calculated by solving one to two real-valued generalized linear fractional programming problems.

Theorem 1.

1. *(Lower bound) Let*

$$f_1 := \inf \lambda \text{ subject to } \underline{A}x \leqslant \lambda\underline{B}x,\ \lambda \leqslant 0,\ \underline{C}x \leqslant \bar{c},\ x \geqslant 0. \quad (2)$$

If $f_1 < 0$ then $\underline{f} = f_1$, otherwise $\underline{f} = f_2$ with

$$f_2 := \inf \lambda \text{ subject to } \underline{A}x \leqslant \lambda\overline{B}x,\ \lambda \geqslant 0,\ \underline{C}x \leqslant \bar{c},\ x \geqslant 0.$$

2. *(Upper bound) Let*

$$f_3 := \inf \lambda \text{ subject to } \overline{A}x \leqslant \lambda\underline{B}x,\ \lambda \geqslant 0,\ \overline{C}x \leqslant \underline{c},\ x \geqslant 0. \quad (3)$$

If $f_3 > 0$ then $\bar{f} = f_3$, otherwise $\bar{f} = f_4$ with

$$f_4 := \inf \lambda \text{ subject to } \overline{A}x \leqslant \lambda\overline{B}x,\ \lambda \leqslant 0,\ \overline{C}x \leqslant \underline{c},\ x \geqslant 0. \quad (4)$$

Proof. 1. (Lower bound) First we consider the case when $\underline{f} < 0$. There is at least one instance of (1) with negative optimal value, so we can restrict our considerations to the family

$$\inf \lambda \text{ subject to } Ax \leqslant \lambda Bx,\ \lambda \leqslant 0,\ Cx \leqslant c,\ x \geqslant 0 \quad (5)$$

with $A \in \mathbf{A},\ B \in \mathbf{B},\ C \in \mathbf{C},\ c \in \mathbf{c}$. For any instance and any feasible point $\lambda,\ x$ we have $\underline{A}x \leqslant Ax \leqslant \lambda Bx \leqslant \lambda\underline{B}x$, and $\underline{C}x \leqslant Cx \leqslant c \leqslant \bar{c}$. It means that $\lambda,\ x$ is also a feasible solution to the problem

$$\inf \lambda \text{ subject to } \underline{A}x \leqslant \lambda\underline{B}x,\ \lambda \leqslant 0,\ \underline{C}x \leqslant \bar{c},\ x \geqslant 0. \quad (6)$$

That is, the feasible set to (6) covers feasible sets of all instances of problems (5). Therefore the lower bound $\underline{f}$ will be achieved for this instance.

Suppose now that $\underline{f} \geqslant 0$. In this case, all instances of (1) have non-negative optimal values, and all their feasible solutions $\lambda,\ x$ have $\lambda \geqslant 0$. That is why (2) is either infeasible or its optimal value is zero. So it remains to show that $\underline{f} \geqslant 0$ implies $\underline{f} = f_2$. Herein, (1) takes the equivalent form

$$\inf \lambda \text{ subject to } Ax \leqslant \lambda Bx,\ \lambda \geqslant 0,\ Cx \leqslant c,\ x \geqslant 0.$$

For any instance and every feasible solution λ, x we have $\underline{A}x \leqslant Ax \leqslant \lambda Bx \leqslant \lambda\overline{B}x$, and $\underline{C}x \leqslant Cx \leqslant c \leqslant \bar{c}$. It means, $\lambda,\ x$ is also a feasible solution to the problem

$$\inf \lambda \text{ subject to } \underline{A}x \leqslant \lambda\overline{B}x,\ \lambda \geqslant 0,\ \underline{C}x \leqslant \bar{c},\ x \geqslant 0. \quad (7)$$

Hence the feasible set to (7) covers feasible sets of all instances of problems (1), and the lower bound $\underline{f}$ will be achieved for this instance.

2. (Upper bound) First we assume that $\bar{f} > 0$. Then there is at least one instance of (1) with positive optimal value, so we can restrict our considerations to the family

$$\inf \lambda \text{ subject to } Ax \leqslant \lambda Bx,\ \lambda \geqslant 0,\ Cx \leqslant c,\ x \geqslant 0 \quad (8)$$

with $A \in \mathbf{A},\ B \in \mathbf{B},\ C \in \mathbf{C},\ c \in \mathbf{c}$. If (3) is infeasible then $\bar{f} = f_3 = \infty$ and we are finished. So let $\lambda,\ x$ be any feasible solution of (3). Then for any instance of (8) we have $Ax \leqslant \overline{A}x \leqslant \lambda\underline{B}x \leqslant \lambda Bx$, and $Cx \leqslant \overline{C}x \leqslant \underline{c} \leqslant c$. Thus $\lambda,\ x$ is a feasible solution to any instance of (8). In other words, the feasible set to (3) is included in a feasible set of any instance of (8). Therefore the highest optimal value will be achieved for $A = \overline{A},\ B = \underline{B},\ C = \overline{C},\ c = \underline{c}$.

Suppose now that $\bar{f} \leqslant 0$. In this case, all instances of (1) have non-positive optimal values, and all their feasible solutions $\lambda,\ x$ have $\lambda \leqslant 0$. That is why (3) is either infeasible or its optimal value is zero. It remains to show that $\bar{f} = f_4$. We rewrite (1) in the equivalent form

$$\inf \lambda \text{ subject to } Ax \leqslant \lambda Bx,\ \lambda \leqslant 0,\ Cx \leqslant c,\ x \geqslant 0. \quad (9)$$

Let $\lambda,\ x$ be any feasible solution of (4); if (4) is infeasible then $\bar{f} = f_4 = \infty$ contradicting our assumption. For any instance of (9) we have $Ax \leqslant \overline{A}x \leqslant \lambda\overline{B}x \leqslant \lambda Bx$, and $Cx \leqslant \overline{C}x \leqslant \underline{c} \leqslant c$. Thus the feasible set to (4) is included in the feasible set of any instance of (9). Therefore the highest optimal value will be achieved in the setting $A = \overline{A},\ B = \overline{B},\ C = \overline{C},\ c = \underline{c}$. □

3. Tolerances of variations

In this section we consider the inverse problem to the previous one. We start with some real-valued problem and want to extend the reals to intervals such that the optimal value of all instances ranges in some prescribed bounds. Analogous problems were studied in linear programming [12], but—to the best of our knowledge—no one discussed any nonlinear case.

Similar kinds of problems are called tolerance analysis, and we usually study how much may certain parameters perturb while preserving some characteristics, e.g. optimality of some point or basis. They were dealt with mainly in linear programming [1,2,11,26–28] concerning only selected parameters (in the objective function or in the right-hand side of constraints). Tolerance analysis for all objective function coefficients in multiobjective linear programming was done in [9,10].