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An improved two-step method for solving generalized Nash equilibrium problems

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ABSTRACT

The generalized Nash equilibrium problem (GNEP) is a noncooperative game in which the strategy set of each player, as well as his payoff function, depend on the rival players strategies. As a generalization of the standard Nash equilibrium problem (NEP), the GNEP has recently drawn much attention due to its capability of modeling a number of interesting conflict situations in, for example, an electricity market and an international pollution control. In this paper, we propose an improved two-step (a prediction step and a correction step) method for solving the quasi-variational inequality (QVI) formulation of the GNEP. Per iteration, we first do a projection onto the feasible set defined by the current iterate (prediction) to get a trial point; then, we perform another projection step (correction) to obtain the new iterate. Under certain assumptions, we prove the global convergence of the new algorithm. We also present some numerical results to illustrate the ability of our method, which indicate that our method outperforms the most recent projection-like methods of Zhang et al. (2010).

1. Introduction

The generalized Nash equilibrium problem, GNEP for short, is a model that has been used extensively in many fields and can date back at least to (Debreu, 1952; Arrow and Debreu, 1954), where the authors called it as a social equilibrium (problem) or an abstract economy. From the 1990s on, the GNEP has attracted more and more attention. Robinson (1993a,b) discussed an application of a GNEP in a two-sided game model of combat, where he analyzed the measure of effectiveness via formulating the model as a GNEP. Wei and Smeers (1999) considered a GNEP constructed from a spatial oligopolistic electricity model with Cournot generators and regulated transmission prices, and proposed a variational inequality approach to determine a solution of the model. Hobbs and Pang (2007) treated oligopolistic electricity models with joint constraints by means of linear complementarity formulations and Contreras et al. (2004) solved electrical market games by GNEP formulations. Breton et al. (2006) analyzed the joint implementation mechanism of environmental projects by formulating the model as a GNEP. Pang et al. (2008) formulated a power allocation problem in parallel interference channels as a GNEP.

The GNEP extends the classical Nash equilibrium problem (NEP for short) by assuming that each player's feasible set can depend on the rival players' strategies. It is by now a well-known fact that the NEP, in which each player solves a convex program, can be formulated and solved as a finite-dimensional variational inequality (VI), and there are a lot of computational methods to solve it; see the monographs of Nagurney (1999) and Facchinei and Pang (2003), and the references therein. While we can also reformulate GNEP as a VI, their solution set coincides only under strong assumptions; see Theorem 2.1 in Facchinei et al. (2007). Due to the interdependence of the feasible set, the GNEP can usually be reformulated as a quasi-variational inequality problem (QVI). The connection between the GNEP and the QVI was recognized by Bensoussan (1974) as early as in 1974 who studied these problems with quadratic functionals in a Hilbert space. Harker (1991) considered these problems in Euclidean spaces. Kocvara and Outrata (1995) discussed a class of QVIs with applications to engineering.

While there are a host of numerical methods for solving VI and NEP, there are only a handful of papers that address QVI in finite dimensions. As Pang and Fukushima pointed out in Pang and Fukushima (2005) (see Pang and Fukushima, 2009; for a notation revision), the study of the QVI to date is in its infancy at best. So, computing a generalized Nash equilibrium is a challenging task up-to-date. It is therefore necessary to design efficient computational methods for solving a GNEP, or its QVI formulation:

1. The first class of methods reformulate the QVI, or KKT system of GNEP as an optimization problem, and find the generalized Nash equilibria (GNEs) via solving the resulting optimization

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problem. Using a regularized Nikaido–Isoda-function, von Heusinger and Kanzow (2009) presented three optimization problems related to the generalized Nash equilibrium problem. The first optimization problem, which possesses the good property that its global minima are the solutions of GNEP, is nonsmooth. The second optimization problem is a smooth and constrained one, whose global minima correspond to the so-called normalized Nash-equilibria. And the third one is an unconstrained optimization problem, whose global minima are the normalized Nash equilibria. The same regularized Nikaido–Isoda function technique was also adopted by Panicucci et al. (2009), where the authors considered an equivalent optimization reformulation, i.e., solutions of GNEP coincide with global minima of the optimization problem. A derivative-free descent type method with inexact line search to solve the equivalent optimization problem was presented there, as well as global convergence proof and some promising numerical results. (Fukushima, 2011) presented an incremental penalty method for GNEPs and showed that it can find a GNE under suitable conditions. In Nabetani et al. (2011), Nabetani et al. derived two types of parameterized VIs related to the GNEP, and show that the parameterized VIs inherit the monotonicity properties of the original VI. Under mild constraint qualifications, they proved that the solutions of the parameterized VIs yield all GNEs. Most recently, Kubota and Fukushima (2010) converted GNEPs into optimization problem, where the objective function is the gap function. The difficulty of all these reformulating techniques is that we need to find the global minima of the resulting optimization problem, which is a difficult task, due to the nonconvexity of the optimization problem.

2. The second class of methods are relaxation-type methods, see e.g., Uryasev and Rubinstein (1994), Krawczyk and Uryasev (2000), Contreras et al. (2004) and Krawczyk (2007). Per iteration, the algorithm needs to solve an optimization problem to find a trial point, and then a relaxation step (convex combination of the current iteration and the trial point) is adopted to generate the next iteration. The problem concerns the optimization of the Nikaido–Isoda function, and the parameters for relaxation can be set prior the iteration, or can be found via solving another optimization problem with single variable (optimized step size). Under certain assumptions (see Theorem 3.1 in Krawczyk and Uryasev (2000)), the sequence generated by the algorithm converges to a normalized Nash equilibrium.

3. The third class of methods are Newton-type methods. Several Newton methods for solving GNEP and QVI were proposed in Facchinei et al. (2009), where features were analyzed and range of applicability were compared. The reported computational results show that the number of iterations of their method is much less than the relaxation method in Uryasev and Rubinstein (1994), while there is no comparison on the cputime used by the two methods. Based on a fixed point reformulation of GNEP, von Heusinger et al., (2010) proposed a nonsmooth Newton method for finding an important subclass of GENs and proved the local superlinear convergence of the algorithm, under the constant rank constraint qualification.

4. The last type of methods are projection-like methods. In Zhang et al. (2010), Zhang et al. proposed two projection-like methods for solving the GNEP and its QVI reformulation. Per iteration, it first finds a trial point via projecting onto a simple point (an optimization problem with quadratic objective function), then generates the next iteration via another simple projection. The method is as simple as the relaxation method, and the main differences are in the optimization problems solved in generating the trial points: in the relaxation method, the objective function in the optimization problem solved per iteration is Nikaido–Isoda-function, while in the projection method, it is a (strongly convex) quadratic function. The projection-like method has advantage when the projection is easy to implements, e.g., when the constraint set is the nonnegative orthant of R^n; a box in R^n; or a ball in R^n. Some preliminary numerical results were also reported in Zhang et al. (2010), indicating the ability of projection methods.

Projection methods play an important role in solving convex optimization problems and monotone variational inequality problems. They are advantageous because of easy implementation; especially, for the problems with simple feasible sets. At the same time, projection methods require little storage, and can readily exploit any separable structure in the corresponding mapping or the constrained set of the problem, i.e., they can perform in a parallel way (Bertsekas and Tsitsiklis, 1989). Due to structural and theoretical advantages, various projection-type methods, such as the basic projection algorithm (Goldstein, 1964; Levitin and Polyak, 1966; He et al., 2002; Han and Sun, 2004), the extragradient algorithm and its variants (He, 1997; Korpelevich, 1976; Khobotov, 1987; Solodov and Tseng, 1996; Sun, 1996; Wang et al., 2001), and the hyperplane projection algorithm (Solodov and Svaiter, 1999; Han and Lo, 2002) have been designed to solve different convex optimization problems or monotone variational inequality problems. Interested readers may consult the survey papers (Ferris and Pang, 1997; Noor, 1997) and the monograph by Facchinei and Pang (2003).

In this paper, we present a new projection-like algorithms for solving a GNEP, based on its QVI reformulation. The method performs in a prediction-correction manner: per iteration, it first performs a prediction step to find a suitable trial point by projecting onto the feasible set that is defined at the current iteration point; then, it performs a correction step to generate the next iteration. The profit direction at the correction step is a combination of two profit directions used in the literature (He, 1997), which performs well in numerical experiments. Under suitable conditions on the underlying mappings of the QVI reformulation of the GNEP, we prove the global convergence of the proposed algorithm. We also report some numerical results to demonstrate the efficiency of the algorithm, which are promising. In summary, the contributions of this paper are as follows:

1. We propose a new projection-like algorithms for solving a GNEP, where we use a similar line-search scheme as that in Zhang et al. (2010) to generate a trial point, but using a new profit direction to generate the next iterate.
2. We prove the global convergence of the proposed algorithm under the assumption that the mapping in QVI, reformulation of GNEP, is co-coercive (see Definition 3.2 for its definition). While the assumption of co-coercivity is stronger than monotonicity, the assumption used in Zhang et al. (2010), it can enhance the numerical performance of the method. See the numerical results reported in Section 6.

The rest of this paper is organized as follows. In Section 2, we state the GENP and its QVI formulation formally. In Section 3, we summarize some definitions, properties of the projection operators and necessary preliminary results that will be used in the sequel. In Section 4, we describe our algorithm and analyze its properties. We prove the global convergence of the algorithm in Section 5 and present our numerical results in Section 6, respectively. Finally, we conclude the paper by giving some concluding remarks in Section 7. For concise presentation, all proofs of the lemmas and theorems were put in Appendix. The proof techniques were partly from some classical papers on projection-type methods and partly from Zhang et al. (2010).