European Journal of Operational Research 216 (2012) 638–646

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Whose deletion does not affect your payoff? The difference between the Shapley value, the egalitarian value, the solidarity value, and the Banzhaf value

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ARTICLE INFO

Article history:
Received 28 January 2011
Accepted 10 August 2011
Available online 18 August 2011

Keywords:
Game theory
Axiomatization
Shapley value
Egalitarian value
Solidarity value
Banzhaf value

ABSTRACT

This study provides a unified axiomatic characterization method of one-point solutions for cooperative games with transferable utilities. Any one-point solution that satisfies efficiency, the balanced cycle contributions property (BCC), and the axioms related to invariance under a player deletion is characterized as a corollary of our general result. BCC is a weaker requirement than the well-known balanced contributions property. Any one-point solution that is both symmetric and linear satisfies BCC. The invariance axioms necessitate that the deletion of a specific player from games does not affect the other players' payoffs, and this deletion is different with respect to solutions. As corollaries of the above characterization result, we are able to characterize the well-known one-point solutions, the Shapley, egalitarian, and solidarity values, in a unified manner. We also studied characterizations of an inefficient one-point solution, the Banzhaf value that is a well-known alternative to the Shapley value.

1. Introduction

Consider a situation that is well described by the standard notion of a cooperative game with a transferable utility (TU game) and consider a one-point solution concept that prescribes how players divide the worth of their total cooperation among themselves. We deal with the setting of variable player sets, in particular, with how a player's payoff in one TU game is related to that in another TU game (especially in the case of a subgame of the former game). More specifically, we explore the problem of when a player's payoff is not affected by the elimination of some other player from the original situation.

From cooperative game theory, we know that there are two relevant factors in considering this problem. Suppose N is an initial player set, v is a characteristic function of N, and $k \in N$ is a player who leaves the game. Then, one of the relevant factors is the difference in the worth of the grand coalition, i.e., the difference between the worth of the initial player set $v(N)$ and that after player k leaves, $v(N\backslash\{k\})$. The other relevant factor is how the bargaining power of players in $N\backslash\{k\}$ has been altered by the elimination of player k. For example, when player i makes a large contribution to the coalitions containing k and zero contribution to any other coalition, he is expected to lose considerable bargaining power after k is deleted.

A null player is a player who makes zero contribution to any other coalition. The worth of the total cooperation is unaltered by the elimination of a null player. In addition, most studies have assumed that the bargaining power of the remaining players is not affected by null player deletion. The original work of Shapley (1953) on the axiomatization of the Shapley value, by using the carrier axiom, which requires the same payoff for players before and after a null player's elimination, was later carefully explored by Derks and Haller (1999), who provided a necessary and sufficient condition for a solution concept to satisfy the property.

In this paper, we explore other types of invariance in the payoffs before and after the elimination of a player. Considering the two factors relevant to the payoff changes when a player is removed, we know that the invariance in the payoff occurs only when the following points are true about the deleted player:

- in the original situation where the player was in set N, his payoff was only his marginal contribution, $v(N) - v(N\backslash\{k\})$, and
- in the situation where the player is deleted, the absolute or relative importance or bargaining power of the remaining players that could have been counted by a solution concept were unaltered.

The first condition is naturally obtained when we deal with a solution concept that distributes the worth of the total cooperation among the players. (i.e., when we focus on an "efficient" solution). The second condition implies that the invariance of the *relative* importance or bargaining power of players is strong enough to

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doi:10.1016/j.ejor.2011.08.011

result in the invariance of the payoffs of the remaining players. Thus, it is possible that a player other than a null player exists whose deletion does not affect the payoff of the remaining player. Here, we consider a player who makes a contribution to each coalition in the same manner. Thus, his deletion does not change the relative bargaining power of the other players. However, note that there is some arbitrariness in the meaning of "making a contribution to each coalition in the same manner." To relieve this arbitrariness, we consider two versions of such a player and examine the invariance properties with respect to the elimination of each type of such players. One player is a proportional player, who makes a contribution to each coalition proportional to its worth and the size of the coalition, while the other player is a quasi-proportional player, who also makes a proportional contribution to each coalition but in a way that slightly differs from the manners in which the proportional player contributes.

We first show that there exist solution concepts in a cooperative game that satisfy these two invariance properties. One invariance property is the invariance in payoffs from proportional player deletion, which is satisfied by the egalitarian value or the equal division value assigning an equal division of $v(N)$ to each of the players in N. The second invariance property, i.e., the invariance in payoffs from quasi-proportional player deletion, is satisfied by the solidarity value introduced by Nowak and Radzik (1994), which is similar to the Shapley value but differs in using the average marginal contributions instead of the marginal contributions of a player.[1] Interestingly, although the difference between these two invariance properties lies in the subtle difference in the meaning of "making a contribution to each coalition in the same manner," the solution concepts that satisfy these invariance axioms are quite different.

Next, we attempt to axiomatize these two solution concepts by using the corresponding invariance axioms. In the case of a null player, Derks and Haller (1999) did not succeed in axiomatizing solution concepts by using the invariance in payoff from the deletion of a null player. Recently, Kamijo and Kongo (2010) axiomatized the Shapley value through this property and their newly defined balanced contributions property. The original balanced contributions property (BC) of Myerson (1980) requires that for any two players, the claim of one player against another, measured by a solution concept, should be balanced with the counter claim from the second player against the first. In contrast, the balanced contributions property proposed by Kamijo and Kongo (2010), which is called the balanced cycle contributions property (BCC), does not require that claims between two players be balanced but rather necessitates that claims among all players should be balanced in a cyclical manner; i.e., for any order of players, the sum of the claims from each player against his successor is balanced with the sum of the claims from that player against his predecessor. Kamijo and Kongo (2010) show that the Shapley value is a unique one-point solution concept that is efficient and satisfies BCC and the invariance in payoff from the deletion of a null player.

One merit of using BCC rather than BC is that while the Shapley value is a unique efficient solution concept satisfying BC, there are several solution concepts that satisfy BCC. In fact, we prove in this paper that any one-point solution that is both symmetric and linear satisfies BCC. Thus, both the egalitarian value and the solidarity value satisfy BCC. Moreover, we show that similar to the results of Kamijo and Kongo (2010), the egalitarian value is axiomatized by the efficiency, BCC, and the invariance from a proportional player deletion, and the solidarity value is axiomatized by the first two along with the invariance from a quasi-proportional player deletion. Thus, we provide new axiomatic foundations of the egalitarian value and the solidarity value, respectively.[2] Furthermore, combined with the result of Kamijo and Kongo (2010), we can see that the difference among the three major one-point solution concepts—the Shapley value, the egalitarian value, and the solidarity value—lies in the selection of a player "whose deletion does not affect your payoff."[3]

An alternative to the Shapley value, the Banzhaf value (Banzhaf, 1965; Owen, 1975) is a well-known *inefficient* solution concept in TU games. A number of studies have compared axiomatizations between the Banzhaf and Shapley values (see, e.g., Lehrer, 1988; Haller, 1994; Feltkamp, 1995; Nowak, 1997; Nowak and Radzik, 2000; Alonso-Meijide et al., 2007; Casajus, 2011b). In most of their results, the Banzhaf value is characterized by replacing efficiency in the sets of axioms characterizing the Shapley value with 2-efficiency, which requires that a merger of two players into a single player does not benefit or harm the two players. A similar observation can be made for our axiomatization of the Shapley value. By replacing the efficiency with 2-efficiency and efficiency with respect to one-person games in the axiomatization of the Shapley value by Kamijo and Kongo (2010), the Banzhaf value is characterized. This result allows us to observe that the difference between the Shapley and the Banzhaf values arises from the difference between efficiency-related axioms. Moreover, we show that, in contrast with the axiomatization results for efficient solutions, there are no one-point solutions that satisfy 2-efficiency, efficiency with respect to one-person games, BCC, and the invariance from a proportional or quasi-proportional player deletion.

The paper is organized as follows. In the next section, we provide the definitions and notation. In Section 3, we introduce the axioms related to the invariance under player deletion. In Section 4, we explain BCC introduced by Kamijo and Kongo (2010) and show that both the egalitarian and the solidarity values satisfy it. In Section 5, we present the general axiomatization results, including axiomatizations of the above two values. In Section 6, we present the axiomatization of the Banzhaf value and two impossibility results. Section 7 offers a conclusion.

2. Preliminaries

Let $N \subseteq \mathbb{N}$ be a finite set of *players* and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ be a *characteristic function*. A pair (N, v) is a *cooperative game with transferable utility*, or simply, a *game*. Let Γ be the set of all games and let $|N| = n$, where $|\cdot|$ represents the cardinality of the set. A nonempty subset $S \subseteq N$ is a coalition, and $v(S)$ is the worth of the coalition. For simplicity, we will represent each singleton $\{i\} \subseteq N$ as i when the possibility of confusion does not exist.

A value or one-point solution concept on Γ is a function that associates each game $(N, v) \in \Gamma$ with an n-dimensional vector in $\mathbb{R}^N$. Let φ be a value on Γ. Given a game $(N, v) \in \Gamma$, *two players* $i, j \in N$ *are symmetric* if for any $S \subseteq N \backslash \{i, j\}$, $v(S \cup i) = v(S \cup j)$. A value φ is *symmetric* if for every $i, j \in N$ that are symmetric in $(N, v) \in \Gamma$, $\varphi_i(N, v) = \varphi_j(N, v)$. A value φ is *linear* if for any real value $\alpha, \ \beta \in \mathbb{R}$ and any two games (N, v), $(N, w) \in \Gamma$, $\alpha\varphi(N, v) + \ \beta\varphi(N, w) = \varphi(N, \alpha v + \ \beta w)$, where $(N, \alpha v + \ \beta w)$ is defined as $(\alpha v + \beta w)(S) = \alpha v(S) + \beta w(S)$ for any $S \subseteq N$. A value φ is *efficient* if

[1] It is worth mentioning that in its recent paper Calvo (2008) found the so-called random removal value for NTU games, where the solidarity value turned out to be the resulting value for TU games.

[2] In the literature, several axiomatizations of the egalitarian value are proposed (for the NTU game framework, see Kalai, 1977; Kalai and Samet, 1985; for the TU game framework, see van den Brink, 2007; for the axiomatization of a class of solutions to which the egalitarian value belongs, see van den Brink and Funaki, 2009). The solidarity value is axiomatized by Nowak and Radzik (1994), Casajus (2011a), and Driessen (2010).

[3] These kinds of unified frameworks for the characterizations of solutions are also provided by Gómez-Rúa and Vidal-Puga (2010) for a broad class of values in TU games with coalition structures. The characterizations of several power indices for the simple games discussed in Lorenzo-Freire et al. (2007) can be seen as a unified framework for characterizations, as well.