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Invited Review

Stochastic Data Envelopment Analysis—A review

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ABSTRACT

This paper provides a review of stochastic Data Envelopment Analysis (DEA). We discuss extensions of deterministic DEA in three directions: (i) deviations from the deterministic frontier are modeled as stochastic variables, (ii) random noise in terms of measurement errors, sample noise, and specification errors is made an integral part of the model, and (iii) the frontier is stochastic as is the underlying Production Possibility Set (PPS).

Stochastic DEA utilizes non-parametric convex or conical hull reference technologies based upon axioms from production theory accompanied by a statistical foundation in terms of axioms from statistics or distributional assumptions. The approaches allow for an estimation of stochastic inefficiency compared to a deterministic or a stochastic PPS and for statistical inference while maintaining an axiomatic foundation. Focus is on bridges and differences between approaches within the field of Stochastic DEA including semi-parametric Stochastic Frontier Analysis (SFA) and Chance Constrained DEA (CCDEA).

We argue that statistical inference based upon homogenous bootstrapping in contrast to a management science approach imposes a restrictive structure on inefficiency, which may not facilitate the communication of results of the analysis to decision makers. Semi-parametric SFA and CCDEA differ w.r.t. the modeling of noise and stochastic inefficiency. The two approaches are in spite of the inherent differences shown to be complements in the sense that the stochastic PPSs obtained by the two approaches share basic similarities in the case of one output and multiple inputs. Recent contributions related to (i) disentangling of random noise and random inefficiency and (ii) obtaining smooth shape constrained estimators of the frontier are discussed.

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1. Introduction

Measuring the relative efficiency and ranking productive performance of Decision Making Units (DMUs) were originally the primary purpose of the Data Envelopment Analysis (DEA) proposed in Charnes, Cooper, and Rhodes (1978) and Banker, Charnes, and Cooper (1984). Each DMU is characterized in these models by an input vector that allows for production of a corresponding output vector. The models are based on sets of axioms that characterize unknown Production Possibility Sets (PPSs). The models constitute an axiomatic approach that explicitly states properties of the reference technology used to measure the relative performance of individual DMUs. The axioms are used to define an estimator of the PPS. Using a given data set that satisfies certain data assumptions, see e.g., Charnes, Cooper, and Thrall (1991), it is possible to measure or estimate, e.g., radial inefficiency of a given observed DMU in input and/or output direction. Indeed, the reference technology reflects both the chosen axioms and the set of observed DMUs, using the principle of minimal extrapolation.

DEA was originally developed within the *Management Science* (MS) *framework*, but without any axiomatic consideration concerning distributional characteristics of the deviation of inefficient DMUs from the best practice frontier and without any specification of noise, i.e., without consideration for measurement errors, sample noise and specification errors. Any given observed set of DMUs was not seen as the result of some sampling process from a larger population. It is interesting to note that when papers on Stochastic DEA began to appear, they took off in two very different directions. One approach, initiated by Banker (1993) included statistical axioms defining a statistical model and a sampling process into the DEA framework. If the analyst accepts these rather restrictive axioms then DEA provides a consistent but biased estimator of the true frontier. Korostelev, Simar, and Tsybakov (1995a, 1995b) proved the consistency of the DEA in a more general setting and provided results on the rate of convergence. Kneip, Park, and Simar (1998) proved consistency in the multivariate setting. Models for approximating sample distributions using bootstrapping procedures were first developed in Simar and Wilson (1998, 1999) to allow for inference on the estimated efficiency

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scores, see also Simar and Wilson (2007) for a related bootstrapping approach that focuses on the impact from environment.

The other approach, initiated by Land, Lovell, and Thore (1993), Olesen and Petersen (1995) and Cooper, Huang, Lelas, Li, and Olesen (1998) (see also Cooper, Huang, & Li, 1996; Olesen, 2006) focused on specifying a random reference technology by replacing the observed input and output data used in a DEA with DMU-specific distributions with supports being subsets of the input output space. Consequently, the performance of the DMUs was not seen as random draws from a common density on the input output space but each DMUs performance was represented by a DMU specific distribution. The theory of chance constraints was used to formalize an efficiency evaluation relative to the random best practice frontier. Some of the advocates of the first approach claim that it is difficult to identify what this second approach is estimating, because no formal statistical model with a sampling process is specified.

The focus in this paper is on a number of different approaches to Stochastic DEA. By Stochastic DEA we mean an efficiency analysis using non-parametric convex hull/convex cone reference technologies based on either statistical axioms or distributional assumptions that allow for a random (estimator of the) reference technology.[1] As noted by Banker (1996), the original DEA formulations assume that the included inputs and outputs are measured without noise, and do not forward any axioms on the distributional structure of deviations from a best practice frontier. The classical DEA models (e.g., the CCR-model and the BCC-model) can be interpreted as providing a deterministic frontier and are for that reason often denoted deterministic. By maintaining two additional statistical axioms Banker (1993) provides an extended environment for DEA that indeed allows for inference. However, this statistical framework comes at a cost[2] which will be discussed in details in this review.

In this paper we view and interpret the notion of Stochastic DEA as comprising a number of proposed methodologies that extend the original idea or framework behind DEA in several different directions:

1. The first direction extends DEA to be able to handle estimated deviations from frontier practice as random deviations.
2. The second direction extends DEA to be able to handle random noise in the form of either measurement errors or specification errors.[3]
3. The third direction extends DEA to be able to regard or conceive the PPS as a random PPS, based on the random variation in data.

Extending DEA in the first direction can, under appropriate assumptions (e.g., a statistical model and a sampling process) be handled within *a statistical* (or econometric) *framework*. A statistical framework requires an axiomatic approach to a statistical model including a specification of a sampling procedure sometimes denoted a Data Generating Process (DGP). A given set of data is regarded as a sample from a large population and a set of efficiency scores is considered one out of many possible outcomes. Consistent estimators and inference in the form of, e.g., confidence intervals have high priority.

Banker (1993) shows that DEA (with one output) provides a consistent estimator of the best practice frontier as a piecewise linear monotonically increasing and concave production function, if one is willing to accept the following two additional axioms: (i) the deviations from the frontier are iid distributed on a one-sided support (only negative residuals are allowed in an output oriented model with one output and multiple inputs), and (ii) the corresponding density function is monotonically decreasing in the absolute size of the residuals. Hence, the DEA estimator can provide inference if one is willing to accept that these axioms are reflecting reality. These necessary assumptions are, however, restrictive but convenient in the sense that they allow for consistency of the DEA based estimation. Banker and coauthors have subsequently published a series of papers showing how to use parametric assumptions on the asymptotic distribution of the inefficiency residuals to define statistical tests of hypotheses related to returns to scale, input substitutability and model specification, see Banker (1996), Banker (1989), Banker and Chang (1995), and Banker, Chang, and Sinha (1994). Simar and Wilson (2002) criticize the semi-parametric assumptions used in Banker (1996) and propose a set of related non-parametric tests on returns to scale using a bootstrapping approach. Simar and Wilson (2001) propose a set of related statistical procedures for testing model structures such as (i) possible input or output aggregations, and (ii) the possible presence of irrelevant inputs or outputs in non-parametric efficiency analyses.

The approaches suggested by Banker and coauthors do not include any estimation of *the sampling distributions* of the estimated efficiency scores. This implies that no information can be extracted on the confidence intervals for each estimated efficiency estimator. A significant and important contribution from Simar and Wilson can be found in a series of papers showing how to use various bootstrapping approaches to get approximations of these unknown sampling distributions and extract confidence intervals on the estimated efficiency scores.[4]

The popular *homogeneous bootstrap* proposed in Simar and Wilson (1998, 1999) is another example of a methodology providing inference in a DEA context but at the cost of a similar set of restrictive but convenient assumptions. This bootstrapping approach is relatively easy to use and works well with relatively few observations in a moderate dimensional input output space. It will, however, be argued below that these necessary assumptions in some cases involve unacceptable structure in the sense that the purpose of the efficiency analysis is violated. These assumptions are convenient by adding structure which implies that the analysis requires less data in order to provide estimators with apparent desired properties.

The homogeneous bootstrap in a cross-section setting can be used to approximate the sample noise and its effect on confidence intervals. But in our opinion, in many practical applications it requires controversial assumptions to use this bootstrap, and one could suspect that the high prevalence of usage of the homogeneous bootstrap is related to the fact that the alternative, the use of a heterogeneous bootstrap, see Simar and Wilson (2000), is prohibitive, because the necessary data typically are not available or difficult to collect. It is, perhaps, in some applications tempting to impose the assumption of a common inefficiency distribution because it provides access to "the relatively easy part" of the bootstrapping apparatus and thereby, perhaps, increases the perceived scientific validity of the analysis and its results. If a homogeneous bootstrap is applied out of pure convenience then one may see discrimination (or less discrimination)

[1] We do not consider Imprecise DEA and Fuzzy DEA within the field of Stochastic DEA, since the statistical foundation for these approaches is non-existing. For a recent review of the fuzzy DEA literature, see Hatami-Marbini, Emrouznejad, and Tavana (2011).

[2] In Section 2.1 we provide an illustration of what we denote a MS DEA application. Specifically, we highlight with reference to a specific example inspired from Sherman and Zhu (2006) that if we apply the convenient but not very realistic assumption of a common inefficiency distribution then this will allow us to get some inference and estimate confidence intervals of the efficiency scores using the homogeneous bootstrap. However, this access to the statistical framework jeopardizes a constructive acceptance from the involved DMUs. It is probably not very convincing to argue that an outstanding performance from a bank branch with an excellent manager is "a lucky draw" and a draw from the (wrongly) estimated common inefficiency distribution. Implicitly, this type of argument assumes that bad performance is going to be a possible outcome next year.

[3] Typically, in econometrics there is a distinction between (i) errors in variables and (ii) errors in equations, see, e.g., chapter 9 in Kmenta (1971).

[4] There are a few successful attempts to derive the asymptotic distributions of the DEA efficiency scores, see, e.g., Gijbels, Mammen, Park, and Simar (1997) for the case of one input and one output.