European Journal of Operational Research 251 (2016) 53–63

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Discrete Optimization

# Lifted Euclidean inequalities for the integer single node flow set with upper bounds

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ARTICLE INFO

*Article history:*
Received 7 January 2015
Accepted 27 October 2015
Available online 30 October 2015

*Keywords:*
Valid inequalities
Mixed integer programming
Polyhedral description
Single node flow set

ABSTRACT

In this paper we discuss the polyhedral structure of the integer single node flow set with two possible values for the upper bounds on the arc flows. Such mixed integer sets arise as substructures in complex mixed integer programs for real application problems.

This work builds on results for the integer single node flow polytope with two arcs given by Agra and Constantino, 2006a. Valid inequalities are extended to a new family, the *lifted Euclidean inequalities*, and a complete description of the convex hull is given. All the coefficients of the facet-defining inequalities can be computed in polynomial time.

We report on some computational experimentations for three problems: an inventory distribution problem, a facility location problem and a multi-item production planning model.



## 1. Introduction

The description of the convex hull of elementary mixed integer sets has been useful in the generation of strong valid inequalities for general mixed integer problems. Particular cases of such elementary sets are the Single Node Flow (SNF) sets (see Fig. 1):

$$\{(y,x) \in \mathbb{Z}_+^{|N|} \times \mathbb{R}^{|N|} : \sum_{t\in N} x_t \le (=)(\ge) D, \ell_t y_t \le x_t \le u_t y_t, t \in N\}.$$

These sets are very common structures that occur after the aggregation of variables and/or constraints of more complex fixed charge capacitated network flow sets.

The single node flow sets have been studied for more than three decades. Padberg, Roy, and Wolsey (1985) studied the case where the $y_i$ are binary and the $\ell_j$ are null. They introduced the so called *flow cover inequalities* and showed this class of valid inequalities suffices to describe the convex hull of the feasible set when $u_j = U, \forall j \in N$. The binary case was also studied in Goemans (1985). Roy and Wolsey (1986) derived the so called *generalized flow cover inequalities* and Stallaert (1997) introduced a new class of valid inequalities by complementing binary variables. For a survey on valid inequalities for this and other related sets from the perspective of lifting see Louveaux & Wolsey, 2003. Special cases where considered by several authors. Constantino (1998) describes the convex hull of several related regions, in particular, the integer single node flow set with $U$ a large positive constant, $U > D$. Agra and Constantino (2006b) provide a polyhedral characterization when $\ell_j = L$ and $u_j = U$ for all $j$. In Agra and Doostmohammadi (2013) several inequalities are extended for the case where there is a set-up variable associated to the node itself.

In this paper we describe the convex hull of the integer SNF set with two possible values for the upper bounds on each arc capacity:

$$\mathcal{X} = \{(y,x) \in \mathbb{Z}_+^{|N|} \times \mathbb{R}^{|N|} : \sum_{t\in N} x_t \le D, 0 \le x_t \le a_1 y_t, t \in N_1,$$
$$0 \le x_t \le a_2 y_t, t \in N_2\},$$

where $\{N_1, N_2\}$ define a partition of $N$. We assume that the coefficients $a_1$, $a_2$ and $D$ are positive integers and $D > \max\{a_1, a_2\}$. While in the classical SNF set the $y$ variables are binary, here they are assumed to be integer. Set $\mathcal{X}$ arises as relaxation of several fixed charge capacitated network flow sets when arc capacities may assume one of the two possible values. See Section 4 for several applications.

The description of $\mathcal{P} = conv(\mathcal{X})$ by linear inequalities is obtained from the description of integer single node flow set involving only two arcs,

$$\mathcal{Z} = \{(y_1, y_2, x_1, x_2) : x_1 + x_2 \le D,\ 0 \le x_1 \le a_1 y_1,$$
$$0 \le x_2 \le a_2 y_2,\ y_1, y_2 \text{ integer}\},$$

given in Agra and Constantino (2006a). It has similarities with the description of the convex hull of the integer single node flow set with constant lower and upper bounds (Agra & Constantino, 2006b).

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http://dx.doi.org/10.1016/j.ejor.2015.10.057
0377-2217/

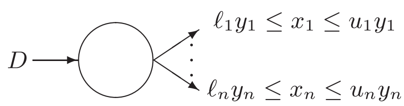


**Fig. 1.** Single node flow problem.

In Section 2 we summarize the results concerned with the description of the SNF problem with two arcs and, in Section 3 we introduce the lifted Euclidean inequalities to generalize those results for the SNF problem with two possible values for the upper bounds. Then, in Section 4 we test the inclusion of those inequalities in a branch and cut scheme to solve three mixed integer programs: an inventory-distribution problem, a facility location problem, and a lot-sizing multi-item problem.

## 2. Euclidean inequalities for the integer single node flow set with two arcs

The results in this section were published in Agra and Constantino (2006a).

First we consider the single node flow set with one arc, $\{(y, x) \in \mathbb{Z}_+ \times \mathbb{R}_+ : x \le D, x \le ay\}$. The convex hull of this set is completely described by the inequalities $x \ge 0$, $x \le ay$, $x \le D$, and $x - \gamma y \le (a - \gamma)\lfloor D/a \rfloor$, where $\gamma = D - a\lfloor D/a \rfloor$. The last inequality is the so-called Mixed Integer Rounding inequality (Nemhauser & Wolsey, 1988).

Next we consider the set with two arcs, $\mathcal{Z}$. It is important to notice that there are only two integer variables involved in this model and so, for this particular structure, all the information needed to describe $conv(\mathcal{Z})$ can also be obtained from the 2-integer knapsack sets that result from the elimination of the continuous variables.

All the extreme points of $conv(\mathcal{Z})$ lie in the intersection of two of the following three hyperplanes defined by $x_1 = a_1 y_1$, $x_2 = a_2 y_2$ and $x_1 + x_2 = D$. Thus, every extreme point of $conv(\mathcal{Z})$ has to satisfy one of the following set of conditions: (i) $x_1 = a_1 y_1$, $x_2 = a_2 y_2$, (ii) $x_2 = a_2 y_2$, $x_1 = D - x_2$, (iii) $x_1 = a_1 y_1$, $x_2 = D - x_1$,

In case (i) we have $(y_1, y_2) \in \mathcal{Y}_\le$ where $\mathcal{Y}_\le = \{(y_1, y_2) \in \mathbb{Z}_+^2 : a_1 y_1 + a_2 y_2 \le D\}$. In case (ii), noticing that $0 \le x_1 \le a_1 y_1$ imply $0 \le D - a_2 y_2 \le a_1 y_1$, we have $(y_1, y_2) \in \mathcal{Y}_1$ where

$$\mathcal{Y}_1 = \{(y_1, y_2) \in \mathbb{Z}_+^2 : a_1 y_1 + a_2 y_2 \ge D, y_2 \le D/a_2\}.$$

Note that constraint $y_2 \le D/a_2$ is implied by the non-negativity constraint $x_1 \ge 0$. Similarly, in case (iii) we have $(y_1, y_2) \in \mathcal{Y}_2$ where

$$\mathcal{Y}_2 = \{(y_1, y_2) \in \mathbb{Z}_+^2 : a_1 y_1 + a_2 y_2 \ge D, y_1 \le D/a_1\}.$$

Let us define $\mathcal{Y}_{1>} = \mathcal{Y}_1 \setminus \mathcal{Y}_=$ and $\mathcal{Y}_{2>} = \mathcal{Y}_2 \setminus \mathcal{Y}_=$ where $\mathcal{Y}_= = \{(y_1, y_2) \in \mathbb{Z}_+^2 : a_1 y_1 + a_2 y_2 = D\}$.

In Agra and Constantino (2006a) it is shown that all the coefficients involved in the computation of the extreme points and facets of the two dimensional polyhedra $conv(\mathcal{Y}_\le)$, $conv(\mathcal{Y}_1)$, $conv(\mathcal{Y}_2)$ can be obtained in $\mathcal{O}(log(D/\min\{a_1, a_2\}))$ elementary operations using a version of the Hirschberg and Wong's algorithm, (Hirschberg & Wong, 1976). This algorithm is based on the Euclidean Algorithm. Hence, the inequalities we describe next, and are based on these two dimensional polyhedra, are referred to as Euclidean inequalities.

First we consider the valid inequalities obtained from the lifting of facet-defining inequalities for $conv(\mathcal{Y}_\le)$ (corresponding to case (i)).

**Proposition 2.1.** *If $\alpha_1 y_1 + \alpha_2 y_2 \le \alpha$ is a valid facet-defining inequality for $conv(\mathcal{Y}_\le)$ then the inequality*

$$\beta_1(x_1 - a_1 y_1) + \beta_2(x_2 - a_2 y_2) + \alpha_1 y_1 + \alpha_2 y_2 \le \alpha \tag{2.1}$$

*is a valid facet-defining inequality for $conv(\mathcal{Z})$, where*

$$\beta_1 = \max\left\{\frac{\alpha_1 y_1 + \alpha_2 y_2 - \alpha}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in \mathcal{Y}_{1>}\right\} \text{ and}$$

$$\beta_2 = \max\left\{\frac{\alpha_1 y_1 + \alpha_2 y_2 - \alpha}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in \mathcal{Y}_{2>}\right\}.$$

Next, from the lifting of the facet defining inequalities of $conv(\mathcal{Y}_1)$ the following family of valid inequalities for $conv(\mathcal{Z})$ is obtained.

**Proposition 2.2.** *If $\alpha_1 y_1 + \alpha_2 y_2 \ge \alpha$ is a valid facet-defining inequality for $conv(\mathcal{Y}_1)$ containing only points in $\mathcal{Y}_{1>}$ then the inequality*

$$\alpha_1 y_1 + \alpha_2 y_2 \ge \alpha + \beta_1(x_1 + x_2 - D) + \beta_2(x_2 - a_2 y_2) \tag{2.2}$$

*is a valid facet-defining inequality for $conv(\mathcal{Z})$, where*

$$\beta_1 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{D - a_1 y_1 - a_2 y_2} : (y_1, y_2) \in \mathcal{Y}_<\right\} \text{ and}$$

$$\beta_2 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in \mathcal{Y}_{2>}\right\}.$$

Finally we consider the lifting of the facet defining inequalities of $conv(\mathcal{Y}_2)$.

**Proposition 2.3.** *If $\alpha_1 y_1 + \alpha_2 y_2 \ge \alpha$ is a valid facet-defining inequality for $conv(\mathcal{Y}_2)$ containing only points in $\mathcal{Y}_{2>}$ the inequality*

$$\alpha_1 y_1 + \alpha_2 y_2 \ge \alpha + \beta_1(x_1 - a_1 y_1) + \beta_2(x_1 + x_2 - D) \tag{2.3}$$

*is a valid facet-defining inequality for $conv(\mathcal{Z})$, where*

$$\beta_1 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{a_1 y_1 + a_2 y_2 - D} : (y_1, y_2) \in \mathcal{Y}_{1>}\right\} \text{ and}$$

$$\beta_2 = \max\left\{\frac{\alpha - \alpha_1 y_1 - \alpha_2 y_2}{D - a_1 y_1 - a_2 y_2} : (y_1, y_2) \in \mathcal{Y}_<\right\}.$$

In Agra and Constantino (2006a) it is shown that the lifting coefficients $\beta_1$ and $\beta_2$ in each Euclidean inequality (2.1), (2.2), (2.3), can be obtained directly (in constant time) from the information required to derive the corresponding two-dimensional polyhedra $conv(\mathcal{Y}_\le)$, $conv(\mathcal{Y}_1)$, $conv(\mathcal{Y}_2)$. So all the coefficients involved in the Euclidean inequalities can be obtained in $\mathcal{O}(log(D/\min\{a_1, a_2\}))$ elementary operations.

Now we consider two unbounded facet-defining inequalities that can be obtained by the MIR procedure.

**Proposition 2.4.** *The inequality*

$$x_t - \gamma_t y_t \le (a_t - \gamma_t)\lfloor D/a_t \rfloor \tag{2.4}$$

*where $\gamma_t = D - a_t\lfloor D/a_t \rfloor$, and $t \in \{1, 2\}$, is valid for $\mathcal{Z}$.*

**Theorem 2.5.** *(Agra and Constantino (2006a)) $conv(\mathcal{Z})$ is completely described by the trivial facet-defining inequalities and the families (2.1), (2.2), (2.3) and (2.4).*

**Example 2.6.** Consider the set, $\mathcal{Z} = \{(x, y) \in \mathbb{R}_+^2 \times \mathbb{Z}_+^2 : x_1 + x_2 \le 1154, \ x_1 \le 21y_1, \ x_2 \le 76y_2\}$ and the following restrictions $\mathcal{Y}_\le = \{y \in \mathbb{Z}_+^2 : 21y_1 + 76y_2 \le 1154\}, \mathcal{Y}_1 = \{y \in \mathbb{Z}_+^2 : 21y_1 + 76y_2 \ge 1154, y_2 \le 15\}, \mathcal{Y}_2 = \{y \in \mathbb{Z}_+^2 : 21y_1 + 76y_2 \ge 1154, y_1 \le 54\}$. The polyhedral description of these sets was given in Agra and Constantino (2006a).

$conv(\mathcal{Y}_\le) = \{y \in \mathbb{R}_+^2 : y_1 + 3y_2 \le 54, \ 2y_1 + 7y_2 \le 109, \ 5y_1 + 18y_2 \le 274, 3y_1 + 11y_2 \le 166, \ y_1 + 4y_2 \le 60\}$. From Proposition 2.1 we obtain the following facet-defining Euclidean inequalities.

$$y_1 + 3y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{14}(x_2 - 76y_2) \le 54$$

$$2y_1 + 7y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{6}(x_2 - 76y_2) \le 109$$

$$5y_1 + 18y_2 + \frac{1}{1}(x_1 - 21y_1) + \frac{1}{3}(x_2 - 76y_2) \le 274$$

$$3y_1 + 11y_2 + \frac{1}{2}(x_1 - 21y_1) + \frac{1}{2}(x_2 - 76y_2) \le 166$$

$$y_1 + 4y_2 + \frac{1}{7}(x_1 - 21y_1) + \frac{1}{7}(x_2 - 76y_2) \le 60$$