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Stronger multi-commodity flow formulations of the (capacitated) sequential ordering problem[☆]

Adam N. Letchford[a], Juan-José Salazar-González[b,*]

[a] Department of Management Science, Lancaster University, Lancaster LA1 4YW, United Kingdom
[b] DMEIO, Facultad de Ciencias, Universidad de La Laguna, 38271 Tenerife, Spain

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ABSTRACT

The *sequential ordering problem* (SOP) is the generalisation of the asymmetric travelling salesman problem in which there are precedence relations between pairs of nodes. Hernández & Salazar introduced a *multi-commodity flow* (MCF) formulation for a generalisation of the SOP in which the vehicle has a limited capacity. We strengthen this MCF formulation by fixing variables and adding valid equations. We then use polyhedral projection, together with some known results on flows, cuts and metrics, to derive new families of strong valid inequalities for both problems. Finally, we give computational results, which show that our findings yield good lower bounds in practice.

1. Introduction

The *sequential ordering problem* (SOP), also called the *asymmetric traveling salesman problem with precedence constraints*, is defined as follows (Escudero, 1988). We are given a directed graph $G = (V, A)$ with $V = \{1, \ldots, n\}$, and a cost c_a for each arc $a \in A$. Node 1 is the *start* node and node n is the *end* node. We are also given an acyclic *precedence digraph*, $H = (V, B)$. The task is to find a minimum-cost Hamiltonian path, starting at node 1 and ending at node n, which obeys the precedences. That is, if $(i, j) \in B$ then i must be visited before j along the path.

The SOP can be used to model vehicle routing problems with pickups and deliveries, and also single-machine scheduling problems with set-up costs and precedences between jobs.

The standard integer programming formulation of the SOP uses binary variables x_a for each $a \in A$, taking the value 1 if and only if arc a is used in the path. For this formulation, many classes of strong valid linear inequalities have been discovered, which have formed the basis of successful exact algorithms for the SOP (e.g., Ascheuer, Escudero, Grötschel, & Stoer, 1993; Ascheuer, Jünger, & Reinelt, 2000; Balas, Fischetti, & Pulleyblank, 1995; Escudero, Guignard, & Malik, 1994). There are also a few papers that discuss alternative formulations that use additional variables, together with appropriate linking constraints (Gouveia & Pesneau, 2006; Gouveia & Ruthmair, 2015; Hernández-Pérez & Salazar-González, 2009).

The present paper was inspired by two existing papers:

- Hernández-Pérez and Salazar-González (2009) presented a *multi-commodity flow* (MCF) formulation for the SOP, and also for a capacitated version of the SOP, called the *multi-commodity one-to-one pickup-and-delivery traveling salesman problem*. (For brevity, we just call it the *CSOP*.)
- Letchford and Salazar-González (2015) presented some new MCF formulations for the so-called *capacitated vehicle routing problem* (CVRP), and showed that they dominate all existing ones, in the sense that their continuous relaxations yield stronger lower bounds.

This paper is concerned with MCF formulations for the SOP and CSOP. As well as presenting new and stronger MCF formulations for both problems, we use some known results on flows, cuts and metrics to project the continuous relaxations of our formulations onto the space of the x variables mentioned above. This yields huge new families of strong valid inequalities for both problems, which include some known inequalities as special cases. We also present computational results, showing that the strengthened MCF formulations yield tight lower bounds in practice as well as in theory.

The paper is structured as follows. Section 2 contains a literature review. Section 3 presents and analyses two simple MCF formulations of the SOP, and Section 4 does the same for two stronger MCF formulations. Section 5 extends the results to the CSOP. Computational

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* Corresponding author. Tel.: +34 922 318184; fax: +34 922 318170.
E-mail addresses: A.N.Letchford@lancaster.ac.uk (A.N. Letchford), jjsalaza@ull.es (J.-J. Salazar-González).

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results are given in Section 6, and some concluding remarks are given in Section 7.

We use the following notation and conventions in the paper. We assume without loss of generality that the arcs $(1, i)$ and (i, n) are in B for all $i \in V \backslash \{1, n\}$. These arcs are called *dummy precedences* while the other arcs in B are called *genuine precedences*. For any $i \in V$, $\pi(i)$ and $\sigma(i)$ denote the *predecessors* and *successors* of i, respectively. That is, $\pi(i)$ and $\sigma(i)$ denote the set of nodes that must be visited before or after i, respectively. We let $H^+ = (V, B^+)$ denote the *transitive closure* of H. That is, $(i, j) \in B^+$ if and only if $i \in \pi(j)$. We also let $H^- = (V, B^-)$ denote the graph obtained from H^+ by deleting all arcs that can be inferred from transitivity. That is, $(i, j) \in B^-$ if and only if $(i, j) \in B^+$ and $\sigma(i) \cap \pi(j) = \emptyset$. Given disjoint sets $S_1, \ldots S_k \subset V$, we let $A(S_1, \ldots S_k)$ denote the set of arcs $(i, j) \in A$ such that there exist integers r, s, with $1 \le r < s \le k$, such that $i \in S_r$ and $j \in S_s$. We define $B^-(S_1, \ldots S_k)$ and $B^+(S_1, \ldots S_k)$ similarly. Given a set $A' \subset A$, $x(A')$ denotes $\sum_{a \in A'} x_a$, and similarly for $u(A')$, $\ell(A')$, etc. Finally, directed graphs, cuts and paths are called *digraphs*, *dicuts* and *dipaths*, respectively.

2. Literature review

We now review the relevant literature. The SOP and CSOP are covered in Sections 2.1 and 2.2, respectively. In Section 2.3, we recall some relevant facts about flows, cuts and metrics.

2.1. The sequential ordering problem

Many formulations and algorithms have been proposed for the SOP (e.g., Ascheuer et al., 1993; Ascheuer et al., 2000; Balas et al., 1995; Bianco, Mingozzi, & Ricciardelli, 1994; Escudero, 1988; Escudero et al., 1994; Gouveia & Pesneau, 2006; Hernández-Pérez & Salazar-González, 2009). The standard integer programming formulation (Ascheuer et al., 1993; Balas et al., 1995; Escudero et al., 1994) uses one binary variable x_a for each arc $a \in A$, taking the value 1 if and only if arc a is traversed in the solution. The formulation takes the form:

$$\min \sum_{a \in A} c_a x_a \tag{1}$$

$$\text{s.t. } x(A(\{i\}, V \setminus \{i\})) = 1 \quad \forall i \in V \setminus \{n\} \tag{2}$$

$$x(A(V \setminus \{i\}, \{i\})) = 1 \quad \forall i \in V \setminus \{1\} \tag{3}$$

$$x(A(S, V \setminus S)) \ge 1 \quad \forall S \subset V \setminus \{1, n\} : S \neq \emptyset \tag{4}$$

$$x(A(S, V \setminus S)) \ge x(A(\{p\}, V \setminus S)) + x(A(\{q\}, S)) \quad \forall S \subset V \setminus \{1, n\}, (p, q) \in B^-(S, V \setminus S) \tag{5}$$

$$x_a \in \{0, 1\} \forall a \in A. \tag{6}$$

Constraints (4) and (5) are called *subtour elimination* (SE) and *precedence-forcing* (PF) inequalities, respectively. Although they are exponential in number, the associated separation problems can be solved efficiently (Ascheuer et al., 1993). (To see that the PF inequalities prevent tours that violate the precedences, consider any arc $(p, q) \in B^-$ and any invalid tour that visits q, followed by a set of nodes T, followed by p. This tour violates the PF inequality with $S = T \cup \{p\}$.)

The polytope associated with the above formulation has been studied in depth (Ascheuer et al., 1993; Ascheuer et al., 2000; Balas et al., 1995; Escudero et al., 1994; Gouveia & Ruthmair, 2015; Mak & Ernst, 2007). Of particular interest to us will be the following four families of valid inequalities:

- The *simple* (π, σ) inequalities (Balas et al., 1995):

$$x(\, A(\, S \setminus (\pi(p) \cup \sigma(q)),\ V \setminus (S \cup \pi(p) \cup \sigma(q))\,)\,) \ge 1 \quad \forall S \subset V \setminus \{1, n\},\ (p, q) \in B^-(S, V \setminus S). \tag{7}$$

- The *precedence-cycle-breaking* (PCB) inequalities (Balas et al., 1995):

$$\sum_{r=1}^{k} x(A(S_r, V \setminus S_r)) \ge k + 1, \tag{8}$$

where $k \ge 2$ is a positive integer, and $S_1, \ldots, S_k$ are disjoint subsets of $V \backslash \{1, n\}$ such that $B^+(S_r, S_{r+1}) \neq \emptyset$ for $r = 1, \ldots, k$, with $S_{k+1} = S_1$.

- The following inequalities, given in Proposition 3.7 of Balas et al. (1995), which we call *P-to-Q* inequalities:

$$x(A(P, Q, V \setminus (Q \cup P))) \ge 2 \quad \forall P, Q \subset V \setminus \{1, n\} : P \cap Q = \emptyset, B^+(P, Q) \neq \emptyset. \tag{9}$$

- The following inequalities, given in Gouveia and Ruthmair (2015), which we call *odd dipath* inequalities:

$$x(A(S, V \setminus S)) \ge \lceil k(S)/2 \rceil, \tag{10}$$

where $S \subset V \backslash \{1, n\}$ and $k(S)$ is the largest integer k such that there exists a dipath $v_0, \ldots, v_k$ in H^- with $v_i \in S$ if and only if i is odd.

Balas et al. (1995) show that the simple (π, σ) inequalities (7) dominate the SE and PF inequalities, yet can be separated in polynomial time. They also point out that each PF inequality (5) is dominated by two P-to-Q inequalities: one in which $P = S$ and $Q = \{q\}$, and the other in which $P = \{p\}$ and $Q = S \cup \{q\} \setminus \{p\}$. Gouveia and Ruthmair (2015) observe that the odd dipath inequalities dominate the SE inequalities.

It is noted in Ascheuer et al. (1993), Balas et al. (1995), Escudero (1988) that certain arcs can be deleted from A without losing any feasible solutions. In our notation, we can delete the arc (i, j) from A if $(j, i) \in B^+$ or if $(i, j) \in B^+ \setminus B^-$. In particular, the arcs entering 1 and leaving n can be deleted, the arc $(1, j)$ can be deleted if $\pi(j) \backslash \{1\} \neq \emptyset$, and the arc (i, n) can be deleted if $\sigma(i) \backslash \{n\} \neq \emptyset$. We assume from now on that all such arcs have been deleted from A.

2.2. The CSOP

As mentioned in the introduction, the CSOP was introduced in Hernández-Pérez and Salazar-González (2009). In the CSOP, the vehicle has a (positive integer) capacity Q and each precedence relation $(p, q) \in B$ is associated with a commodity that has a weight of d_{pq} and needs to be collected at p and delivered at q. (We remark that a relaxed version of the CSOP, in which the vehicle is permitted to visit nodes more than once, was presented in an earlier paper Timlin and Pulleyblank, 1992.)

The following MCF formulation of the CSOP was presented in Hernández-Pérez and Salazar-González (2009). For each $a \in A$ and each $b \in B$, define the binary variable f_a^b, taking the value 1 if and only if commodity b is carried across arc a. Then take (1)–(3) and (6), and add the following constraints:

$$f^b(A(\{i\}, V \setminus \{i\})) - f^b(A(V \setminus \{i\}, \{i\})) = d_i^b \quad \forall i \in V,\ b \in B \tag{11}$$

$$0 \le f_a^b \le x_a \quad \forall a \in A, b \in B \tag{12}$$

$$\textstyle\sum_{b \in B} d_b f_a^b \le Q x_a \quad \forall a \in A, \tag{13}$$

where the constant d_i^b takes the value 1 if i is the origin of the commodity b, -1 if i is the destination of b, and 0 otherwise.

The following enhancements to this model were also proposed in Hernández-Pérez and Salazar-González (2009):

- One can change the first inequality in (12) to an equation if (i) the head of a is the tail of b, (ii) the head of b is the tail of a, (iii) the tail of a is 1 but the tail of b is not, or (iv) the head of a is n but the head of b is not. One can also change the second inequality in (12) to an equation if a and b share a common head or tail.